A TABLEAU FORMULA FOR ETA POLYNOMIALS

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Abstract. We use the Pieri and Giambelli formulas of [BKT1, BKT3] and the calculus of raising operators developed in [BKT2, T1] to prove a tableau formula for the eta polynomials of [BKT3] and the Stanley symmetric functions which correspond to Grassmannian elements of the Weyl group $\widetilde{W}_n$ of type D_n. We define the *skew elements* of $\widetilde{W}_n$ and exhibit a bijection between the set of reduced words for any skew $w \in \widetilde{W}_n$ and a set of certain standard typed tableaux on a skew shape λ/μ associated to w.

1. Introduction

Let k and n be positive integers with $k \leq n$ and $\mathrm{OG} = \mathrm{OG}(n+1-k, 2n+2)$ be the even orthogonal Grassmannian which parametrizes isotropic subspaces of dimension $n+1-k$ in a complex vector space of dimension $2n+2$, equipped with a nondegenerate symmetric bilinear form. Following [BKT1], the Schubert classes σ_λ in $\mathrm{H}^*(\mathrm{OG}, \mathbb{Z})$ are indexed by *typed k-strict partitions* λ. Recall that a k-strict partition is an integer partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that all parts λ_i greater than k are distinct. A typed k-strict partition is a pair consisting of a k-strict partition λ together with an integer $\mathrm{type}(\lambda) \in \{0, 1, 2\}$ which is positive if and only if $\lambda_i = k$ for some index i.

In [BKT3], we discovered a remarkable connection between the cohomology of even (type D) and odd (type B) orthogonal Grassmannians, which allows a uniform approach to the Schubert calculus on both varieties. We used this together with our earlier work [BKT2] to obtain a *Giambelli formula* for the Schubert class σ_λ, which expresses it as a polynomial in certain special Schubert classes. This polynomial, which is called an *eta polynomial*, and denoted H_λ, is defined using Young's raising operators [Li, Y]. These Giambelli polynomials naturally live in the stable cohomology ring of OG, and multiply like the Schubert classes on $\mathrm{OG}(n+1-k, 2n+2)$ when n is sufficiently large.

The present paper is concerned with the principal specialization $H_\lambda(x\,;y)$ of H_λ in the ring of type D Billey-Haiman Schubert polynomials [BH]. In [BKT3], we proved that the polynomial $H_\lambda(x\,;y)$ may be identified with the Billey-Haiman Schubert polynomial $\mathfrak{D}_{w_\lambda}(x\,;y)$ indexed by the corresponding k-Grassmannian element w_λ in the Weyl group $\widetilde{W}_{n+1}$ for the root system of type D_{n+1}. Note that $\mathfrak{D}_{w_\lambda}(x\,;y)$ is really a formal power series, and that its equality with $H_\lambda(x\,;y)$ holds only modulo the relations among the Schur P-functions which enter into the definition of $H_\lambda(x\,;y)$ (see §4.2 for more details). A significant application of these eta polynomials had appeared earlier in [T2, §6], where they were used in splitting

Date: November 18, 2013.
2000 *Mathematics Subject Classification.* Primary 05E15; Secondary 14M15, 14N15, 05E05.
The author was supported in part by NSF Grants DMS-0901341 and DMS-1303352.

formulas for the general type D Schubert polynomials $\mathfrak{D}_w(x\,;y)$. This result was applied in [T2] to prove combinatorially explicit Chern class formulas for degeneracy loci of vector bundles in the sense of Fulton [Fu]. The reader may consult [T3] for an exposition of this work, which covers all the classical Lie groups.

The first goal of this article is to combine the above algebraic and combinatorial theory with the raising operator approach to tableau formulas developed in [T1]. The constructions here are analogous to the type C case studied in op. cit., but we take the opportunity to clarify and simplify some of the earlier proofs, when treating the corresponding results (especially Theorem 1 and Theorem 4).

Our main theorem (Theorem 3) is a formula for the eta polynomial $H_\lambda(x\,;y)$, which writes it as a sum of monomials $(xy)^U$ over certain fillings U of the Young diagram of λ called *typed k'-bitableaux*. The analysis turns out to be rather subtle, since a direct synthesis of the results of [BKT3] and [T1] does not lead to *positive* formulas (compare with [BKT3, Prop. 5.4]). We obtain the key reduction formula for H_λ (Theorem 2) by combining the reduction formulas for $\widehat{H}_\lambda$ and $\widetilde{H}_\lambda$, which are polynomial constituents of H_λ. It is not a priori clear that such an approach will work; the resulting story is the main innovation of this paper.

We call an element w of $\widetilde{W}_{n+1}$ *skew* if there exists a $w' \in \widetilde{W}_{n+1}$ and a k-Grassmannian element w_λ such that $ww' = w_\lambda$ and $\ell(w) + \ell(w') = |\lambda|$. The skew elements of the symmetric group are precisely the 321-avoiding permutations or fully commutative elements, which were introduced in [BJS] and explored further in [S1, S2]. In the other classical Lie types, although every fully commutative element is skew, the converse is false. The skew elements of the the hyperoctahedral group were studied in [T1], and we extend this theory here to $\widetilde{W}_{n+1}$.

Let λ and μ be typed k-strict partitions such that the diagram of μ is contained in the diagram of λ. Our approach leads naturally to the definition of certain symmetric functions $E_{\lambda/\mu}(x)$, given as a sum of monomials x^T corresponding to *typed k'-tableaux* T on the skew shape λ/μ. We find that the function $E_{\lambda/\mu}(x)$, when non-zero, is equal to a type D Stanley symmetric function $E_w(x)$ indexed by a skew element w in $\widetilde{W}_{n+1}$. As in [BJS] and [T1], there is an explicit bijection between the standard typed k'-tableaux on the skew shape λ/μ and the reduced words for a corresponding skew element $w \in \widetilde{W}_{n+1}$.

This paper is organized as follows. Section 2 reviews the Giambelli and Pieri formulas which hold in the Chern subring $\Omega^{(k)}$ of the stable cohomology ring of $\mathrm{OG}(n+1-k, 2n+2)$ as $n \to \infty$. We also establish the *mirror identity* in $\Omega^{(k)}$, a key technical tool which provides a bridge between the Pieri rule and our tableau formulas. Section 3 introduces the eta polynomials and proves various reduction formulas which are then combined to obtain our main tableau formula for $H_\lambda(x\,;y)$. Finally, in Section 4 we relate this theory to the type D Schubert polynomials and Stanley symmetric functions of [BH, L], and study the skew elements of $\widetilde{W}_{n+1}$.

The author is grateful to the referee for a careful reading of the paper and for suggestions which helped to improve the exposition.

2. Preliminary results

2.1. **Raising operators.** An *integer sequence* is a sequence of integers $\alpha = \{\alpha_i\}_{i\geq 1}$, only finitely many of which are non-zero. The *length* of α, denoted $\ell(\alpha)$, is the largest integer $\ell \geq 0$ such that $\alpha_\ell \neq 0$. We will identify an integer sequence of length ℓ with the vector consisting of its first ℓ terms. We set $|\alpha| = \sum \alpha_i$ and let

$\#\alpha$ equal the number of non-zero parts α_i of α. The inequality $\alpha \geq \beta$ means that $\alpha_i \geq \beta_i$ for each i. We say that α is a *composition* if $\alpha_i \geq 0$ for all i and a *partition* if $\alpha_i \geq \alpha_{i+1} \geq 0$ for all i. We will represent a partition λ by its Young diagram of boxes, which has λ_i boxes in row i for each $i \geq 1$. The containment relation between two Young diagrams is denoted by $\mu \subset \lambda$ instead of $\mu \leq \lambda$; in this case the set-theoretic difference $\lambda \smallsetminus \mu$ is called a skew diagram and is denoted by λ/μ.

Given any integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $i < j$, we define

$$R_{ij}(\alpha) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots);$$

a raising operator R is any monomial in these R_{ij}'s. Given any formal power series $\sum_{i \geq 0} c_i t^i$ in the variable t and an integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we write $c_\alpha = c_{\alpha_1} c_{\alpha_2} \cdots c_{\alpha_\ell}$ and set $R\, c_\alpha = c_{R\alpha}$ for any raising operator R. We will always work with power series with constant term 1, so that $c_0 = 1$ and $c_i = 0$ for $i < 0$.

2.2. **The Giambelli formula.** Fix an integer $k > 0$. We consider an infinite family $\omega_1, \omega_2, \ldots$ of commuting variables, with ω_i of degree i for all i, and set $\omega_0 = 1$, $\omega_r = 0$ for $r < 0$, and $\omega_\alpha = \prod_i \omega_{\alpha_i}$. Let $I^{(k)} \subset \mathbb{Z}[\omega_1, \omega_2, \ldots]$ be the ideal generated by the relations

$$\frac{1 - R_{12}}{1 + R_{12}}\, \omega_{(r,r)} = \omega_r^2 + 2 \sum_{i=1}^{r} (-1)^i \omega_{r+i} \omega_{r-i} = 0 \quad \text{for } r > k.$$

Define the graded ring $\Omega^{(k)} = \mathbb{Z}[\omega_1, \omega_2, \ldots]/I^{(k)}$.

Let $\Delta^\circ = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i < j\}$ and equip Δ° with the partial order $\leq$ defined by $(i', j') \leq (i, j)$ if and only if $i' \leq i$ and $j' \leq j$. A finite subset D of Δ° is a *valid set of pairs* if it is an order ideal, i.e., $(i,j) \in D$ implies $(i', j') \in D$ for all $(i', j') \in \Delta^\circ$ with $(i', j') \leq (i, j)$. For any valid set of pairs D, we define the raising operator

$$R^D = \prod_{i<j} (1 - R_{ij}) \prod_{i<j \,:\, (i,j) \in D} (1 + R_{ij})^{-1}.$$

A partition λ is *k-strict* if all its parts greater than k are distinct. The *k-length* of λ, denoted $\ell_k(\lambda)$, is the cardinality of the set $\{i \mid \lambda_i > k\}$. We say that λ has *positive type* if $\lambda_i = k$ for some index i. The monomials ω_λ for λ a k-strict partition form a $\mathbb{Z}$-basis of $\Omega^{(k)}$.

Given a k-strict partition λ, we define a valid set of pairs $\mathcal{C}(\lambda)$ by

$$\mathcal{C}(\lambda) = \{(i,j) \in \Delta^\circ \mid \lambda_i + \lambda_j \geq 2k + j - i \text{ and } j \leq \ell(\lambda)\}$$

and set $R^\lambda := R^{\mathcal{C}(\lambda)}$. Furthermore, define $\Omega_\lambda \in \Omega^{(k)}$ by the *Giambelli formula*

$$\Omega_\lambda := R^\lambda\, \omega_\lambda. \tag{1}$$

Since

$$\Omega_\lambda = \omega_\lambda + \sum_{\mu \succ \lambda} b_{\lambda\mu}\, \omega_\mu$$

with the sum over k-strict partitions μ which strictly dominate λ, it follows that the Ω_λ as λ runs over k-strict partitions form another $\mathbb{Z}$-basis of $\Omega^{(k)}$. More generally, given any valid set of pairs D and an integer sequence α, we denote $R^D\, \omega_\alpha$ by Ω^D_α.

Example 1. In the ring $\Omega^{(2)}$ we have

$$\begin{aligned}\Omega_{322} &= \frac{1-R_{12}}{1+R_{12}}(1-R_{13})(1-R_{23})\,\omega_{322} \\ &= (1-2R_{12}+2R_{12}^2-\cdots)(1-R_{13})(1-R_{23})\,\omega_{322} \\ &= (1-2R_{12}+2R_{12}^2-2R_{12}^3)(1-R_{13}-R_{23}+R_{13}R_{23})\,\omega_{322} \\ &= \omega_{322}-\omega_{421}-2\,\omega_7+2\,\omega_{61}-\omega_{331}+\omega_{43} \\ &= \omega_3\,\omega_2^2-\omega_4\,\omega_2\,\omega_1-2\,\omega_7+2\,\omega_6\,\omega_1-\omega_3^2\,\omega_1+\omega_4\,\omega_3.\end{aligned}$$

2.3. **Cohomology of even orthogonal Grassmannians.** Let the vector space $V=\mathbb{C}^{2n+2}$ be equipped with a nondegenerate symmetric bilinear form. A subspace Σ of V is called *isotropic* if the form vanishes when restricted to Σ. The dimensions of such isotropic subspaces Σ range from 0 to $n+1$.

Choose k with $0<k\leq n$ and let $\mathrm{OG}=\mathrm{OG}(n+1-k,2n+2)$ denote the Grassmannian parametrizing isotropic subspaces of V of dimension $n{+}1{-}k$. Let $\mathcal{Q}$ denote the universal quotient vector bundle of rank $n+1+k$ over OG, and $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})_1$ denote the subring of the cohomology ring $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})$ which is generated by the Chern classes of $\mathcal{Q}$. There is a ring epimorphism $\psi:\Omega^{(k)}\to\mathrm{H}^*(\mathrm{OG},\mathbb{Z})_1$ which maps the generators ω_p to the Chern classes $c_p(\mathcal{Q})$ for each $p\geq 1$; in particular we have $\psi(\omega_p)=0$ if $p>n+k$.

It follows from [BKT3, Theorem 1] that for any k-strict partition λ, we have

$$\psi(2^{-\ell_k(\lambda)}\,\Omega_\lambda)=[Y_\lambda]$$

if the diagram of λ fits inside a rectangle $\mathcal{R}$ of size $(n{+}1{-}k)\times(n{+}k)$, and $\psi(\Omega_\lambda)=0$ otherwise. Here Y_λ is a certain Zariski closed subset of pure codimension $|\lambda|$ in OG, which is a Schubert variety in OG, if λ does not have positive type, and a union of two Schubert varieties in OG, if λ has positive type. The cohomology classes $[Y_\lambda]$ for λ a k-strict partition contained in $\mathcal{R}$ form a $\mathbb{Z}$-basis for $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})_1$. More details and the proofs of these facts are provided in [BKT3].

2.4. **The Pieri rule.** We let $[r,c]$ denote the box in row r and column c of a Young diagram. We say that the boxes $[r,c]$ and $[r',c']$ are *k'-related* if

$$\left|c-k-\frac{1}{2}\right|+r=\left|c'-k-\frac{1}{2}\right|+r'.$$

In the diagram of Figure 1, the two grey boxes are k'-related. The definition of k'-related boxes was introduced in [BKT1, §3.2], and is a type D analogue of the notion of k-related boxes, which is used in the Lie types B and C. The equivalence relation 'k'-related' is the same as 'k-related' when k is replaced by the half integer $k-1/2$. We will define 'k-related' precisely in §2.6.

Given two partitions λ and μ with $\lambda\subset\mu$, the skew Young diagram μ/λ is called a *horizontal strip* (respectively, *vertical strip*) if it does not contain two boxes in the same column (respectively, row). For any two k-strict partitions λ and μ, let α_i (respectively β_i) denote the number of boxes of λ (respectively μ) in column i, for $1\leq i\leq k$. We have a relation $\lambda\to\mu$ if μ can be obtained by removing a vertical strip from the first k columns of λ and adding a horizontal strip to the result, so that for each i with $1\leq i\leq k$,

(1) if $\beta_i=\alpha_i$, then the box $[\alpha_i,i]$ is k'-related to at most one box of $\mu\smallsetminus\lambda$; and

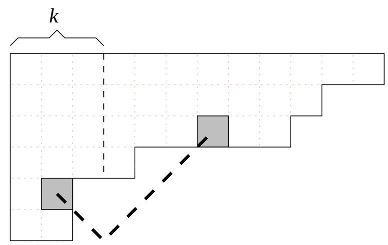


FIGURE 1. Two k'-related boxes in a k-strict Young diagram

(2) if $\beta_i < \alpha_i$, then the boxes $[\beta_i, i], \ldots, [\alpha_i, i]$ must each be k'-related to exactly one box of $\mu \smallsetminus \lambda$, and these boxes of $\mu \smallsetminus \lambda$ must all lie in the same row.

If $\lambda \to \mu$, we let $\mathbb{A}'$ be the set of boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ and higher which are *not* mentioned in (1) or (2). Define the connected components of $\mathbb{A}'$ by agreeing that two boxes in $\mathbb{A}'$ are connected if they share at least a vertex. Then define $N(\lambda, \mu)$ to be the number of connected components of $\mathbb{A}'$, and set

$$M(\lambda,\mu) = \ell_k(\lambda) - \ell_k(\mu) + \begin{cases} N(\lambda,\mu)+1 & \text{if } \lambda \text{ has positive type and } \mu \text{ does not,} \\ N(\lambda,\mu) & \text{otherwise.} \end{cases}$$

We deduce from [BKT3, Theorem 5] and the discussion in §2.3 that the following Pieri rule holds: For any k-strict partition λ and integer $p \geq 0$,

$$\omega_p \cdot \Omega_\lambda = \sum_{\substack{\lambda \to \mu \\ |\mu| = |\lambda| + p}} 2^{M(\lambda,\mu)}\, \Omega_\mu\,. \tag{2}$$

To compare with [BKT3, §1], observe that the notion of K-related boxes used in loc. cit. agrees with the notion of k'-related boxes when the dimension N of the ambient vector space is even (so we are in Lie type D).

For any $d \geq 1$ define the raising operator R^λ_d by

$$R^\lambda_d = \prod_{1 \leq i < j \leq d} (1 - R_{ij}) \prod_{i<j\,:\,(i,j) \in \mathcal{C}(\lambda)} (1 + R_{ij})^{-1}.$$

We compute that

$$\omega_p \cdot \Omega_\lambda = \omega_p \cdot R^\lambda_\ell\, \omega_\lambda = R^\lambda_{\ell+1} \cdot \prod_{i=1}^{\ell} (1 - R_{i,\ell+1})^{-1}\, \omega_{\lambda,p}$$

$$= R^\lambda_{\ell+1} \cdot \prod_{i=1}^{\ell} (1 + R_{i,\ell+1} + R^2_{i,\ell+1} + \cdots)\, \omega_{\lambda,p} = \sum_{\nu \in \mathcal{N}(\lambda,p)} \Omega^{\mathcal{C}(\lambda)}_\nu\,,$$

where $\mathcal{N} = \mathcal{N}(\lambda, p)$ is the set of all compositions $\nu \geq \lambda$ such that $|\nu| = |\lambda| + p$ and $\nu_j = 0$ for $j > \ell + 1$, where ℓ denotes the length of λ. Equation (2) is therefore equivalent to the identity

$$\sum_{\nu \in \mathcal{N}(\lambda,p)} \Omega^{\mathcal{C}(\lambda)}_\nu = \sum_{\substack{\lambda \to \mu \\ |\mu| = |\lambda| + p}} 2^{M(\lambda,\mu)}\, \Omega^{\mathcal{C}(\mu)}_\mu\,, \tag{3}$$

which was proved in [BKT3].

2.5. **The mirror identity.** Let λ and ν be k-strict partitions such that

$$\nu_1 > \max(\lambda_1, \ell(\lambda) + 2k - 1)$$

and choose $p, m \geq 0$. Then the Pieri rule (2) implies that the coefficient of Ω_ν in the Pieri product $\omega_p \cdot \Omega_\lambda$ is equal to the coefficient of $\Omega_{(\nu_1+m,\nu_2,\nu_3,\ldots)}$ in the product $\omega_{p+m} \cdot \Omega_\lambda$. We apply this to make the following important definition.

Definition 1. Let λ and μ be k-strict partitions with $\mu \subset \lambda$, and choose any $p \geq \max(\lambda_1+1, \ell(\lambda)+2k-1)$. If $|\lambda| = |\mu|+r$ and $\lambda \to (p+r, \mu)$, then we write $\mu \rightsquigarrow \lambda$ and say that λ/μ is a *k'-horizontal strip*. We define $m(\lambda/\mu) := M(\lambda, (p+r, \mu))$; in other words, the numbers $m(\lambda/\mu)$ are the exponents that appear in the Pieri product

$$\omega_p \cdot \Omega_\lambda = \sum_{r,\mu} 2^{m(\lambda/\mu)}\, \Omega_{(p+r,\mu)} \tag{4}$$

with the sum over integers $r \geq 0$ and k-strict partitions $\mu \subset \lambda$ with $|\mu| = |\lambda| - r$.

Note that a k'-horizontal strip λ/μ is a pair of partitions λ and μ with $\mu \rightsquigarrow \lambda$. As such it depends on λ and μ and not only on the difference $\lambda \smallsetminus \mu$.

Lemma 1. *Let* $\Psi = \prod_{j=2}^{\ell+1} \frac{1-R_{1j}}{1+R_{1j}}$, *and suppose that we have an equation*

$$\sum_\nu a_\nu \omega_\nu = \sum_\nu b_\nu \omega_\nu$$

in $\Omega^{(k)}$, *where the sums are over all* $\nu = (\nu_1, \ldots, \nu_\ell)$, *while* a_ν *and* b_ν *are integers only finitely many of which are non-zero. Then we have*

$$\sum_\nu a_\nu \Psi\, \omega_{(p,\nu)} = \sum_\nu b_\nu \Psi\, \omega_{(p,\nu)}$$

in the ring $\Omega^{(k)}$, *for any integer* p.

Proof. The proof is the same as [T1, Proposition 2]. □

Let λ be any k-strict partition of length ℓ. Consider the following version of (3):

$$\sum_{\alpha \geq 0} \Omega^{\mathcal{C}(\lambda)}_{\lambda+\alpha} = \sum_{\lambda \to \mu} 2^{M(\lambda,\mu)}\, \Omega^{\mathcal{C}(\mu)}_\mu \tag{5}$$

where the first sum is over all compositions α of length at most $\ell+1$, and the second over k-strict partitions μ with $\lambda \to \mu$. The next result is called the *mirror identity* of (5), and is an analogue of [T1, Theorem 2]; the proof we give below simplifies the one found in loc. cit.

Theorem 1. *For* λ *any* k*-strict partition we have*

$$\sum_{\alpha \geq 0} 2^{\#\alpha}\, \Omega^{\mathcal{C}(\lambda)}_{\lambda-\alpha} = \sum_{\mu \rightsquigarrow \lambda} 2^{m(\lambda/\mu)}\, \Omega_\mu, \tag{6}$$

where the first sum is over all compositions α.

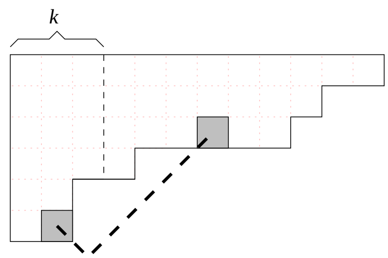


FIGURE 2. Two $(k-1)$-related boxes in a Young diagram

Proof. Choose $p \geq |\lambda| + 2k$ and let $\ell = \ell(\lambda)$. Expanding the Giambelli formula with respect to the first row gives

$$\omega_p \cdot \Omega_\lambda = R^{\mathcal{C}(p,\lambda)} \prod_{j=2}^{\ell+1} \frac{1+R_{1j}}{1-R_{1j}}\, \omega_{(p,\lambda)} = R^{\mathcal{C}(p,\lambda)} \prod_{j=2}^{\ell+1} (1 + 2R_{1j} + 2R_{1j}^2 + \cdots)\, \omega_{(p,\lambda)}$$
$$= \sum_{\alpha \geq 0} 2^{\#\alpha}\, \Omega^{\mathcal{C}(p,\lambda)}_{(p+|\alpha|,\lambda-\alpha)}.$$

Comparing this with (4), we deduce that

$$\sum_{\alpha\geq 0} 2^{\#\alpha}\, \Omega^{\mathcal{C}(p,\lambda)}_{(p+|\alpha|,\lambda-\alpha)} = \sum_{\mu \rightsquigarrow \lambda} 2^{m(\lambda/\mu)}\, \Omega_{(p+|\lambda|-|\mu|,\mu)}.$$

We claim that for every integer $r \geq 0$,

$$\sum_{|\alpha|=r} 2^{\#\alpha}\, \Omega^{\mathcal{C}(p,\lambda)}_{(p+r,\lambda-\alpha)} = \sum_{\substack{\mu\rightsquigarrow\lambda \\ |\mu|=|\lambda|-r}} 2^{m(\lambda/\mu)}\, \Omega_{(p+r,\mu)}. \tag{7}$$

The proof is by induction on r. The base case $r = 0$ is clearly true. For the induction step, suppose that we have for some $r > 0$ that

$$\sum_{s\geq r}\sum_{|\alpha|=s} 2^{\#\alpha}\, \Omega^{\mathcal{C}(p,\lambda)}_{(p+s,\lambda-\alpha)} = \sum_{s\geq r}\sum_{\substack{\mu\rightsquigarrow\lambda \\ |\mu|=|\lambda|-s}} 2^{m(\lambda/\mu)}\, \Omega_{(p+s,\mu)}. \tag{8}$$

Expanding the Giambelli formula with respect to the first component, we obtain $\omega_{p+s}\, \Omega^{\mathcal{C}(\lambda)}_{\lambda-\alpha}$ as the leading term of $\Omega^{\mathcal{C}(p,\lambda)}_{(p+s,\lambda-\alpha)}$, while $\omega_{p+s}\, \Omega_\mu$ is the leading term of $\Omega_{(p+s,\mu)}$. Since the set of all products $\omega_d\, \Omega_\nu$ for which (d,ν) is a k-strict partition is linearly independent in $\Omega^{(k)}$, we deduce from (8) that

$$\sum_{|\alpha|=r} 2^{\#\alpha}\, \Omega^{\mathcal{C}(\lambda)}_{\lambda-\alpha} = \sum_{\substack{\mu\rightsquigarrow\lambda \\ |\mu|=|\lambda|-r}} 2^{m(\lambda/\mu)}\, \Omega_\mu. \tag{9}$$

By applying Lemma 1 to (9), we see that (7) is true, and this completes the induction. This also finishes the proof of Theorem 1, since the argument shows that (9) holds for every integer $r \geq 0$. □

2.6. **The set $\mathbb{A}$ and the integer $N(\mathbb{A})$.** We say that boxes $[r,c]$ and $[r',c']$ are $(k-1)$-*related* if

$$|c-k| + r = |c'-k| + r'.$$

For example, the two grey boxes in the diagram of Figure 2 are $(k-1)$-related. We call box $[r,c]$ a *left box* if $c \leq k$ and a *right box* if $c > k$.

If $\mu \subset \lambda$ are two k-strict partitions such that λ/μ is a k'-horizontal strip, we define $\lambda_0 = \mu_0 = \infty$ and agree that the diagrams of λ and μ include all boxes $[0, c]$ in row zero. We let R (respectively $\mathbb{A}$) denote the set of right boxes of μ (including boxes in row zero) which are bottom boxes of λ in their column and are (respectively are not) $(k-1)$-related to a left box of λ/μ. Let $N(\mathbb{A})$ denote the number of connected components of $\mathbb{A}$. Moreover, define

$$\widehat{n}(\lambda/\mu) = \ell_k(\mu) - \ell_k(\lambda) + m(\lambda/\mu).$$

Lemma 2. *A pair $\mu \subset \lambda$ of k-strict partitions forms a k'-horizontal strip λ/μ if and only if (i) λ/μ is contained in the rim of λ, and the right boxes of λ/μ form a horizontal strip; (ii) no two boxes in R are $(k-1)$-related; and (iii) if two boxes of λ/μ lie in the same column, then they are $(k-1)$-related to exactly two boxes of R, which both lie in the same row. We have*

$$\widehat{n}(\lambda/\mu) = \begin{cases} N(\mathbb{A}) & \text{if } \lambda \text{ has positive type and } \mu \text{ does not,} \\ N(\mathbb{A}) - 1 & \text{otherwise.} \end{cases} \tag{10}$$

Proof. We have $\mu \rightsquigarrow \lambda$ if and only if $\lambda \to (p+r, \mu)$ for any $p \geq |\lambda| + 2k$, where $r = |\lambda - \mu|$. Observe that a box of $(p+r, \mu) \smallsetminus \lambda$ corresponds to a box of μ which is a bottom box of λ in its column. The rest of the proof is a straightforward translation of the definitions in §2.4. □

3. Reduction formulas and tableaux

3.1. **Schur and theta polynomials.** Let $x = (x_1, x_2, \ldots)$ and set $y = (y_1, \ldots, y_k)$ for a fixed integer $k \geq 1$. Consider the generating series

$$\prod_{i=1}^{\infty} \frac{1 + x_i t}{1 - x_i t} = \sum_{r=0}^{\infty} q_r(x) t^r \quad \text{and} \quad \prod_{j=1}^{k} (1 + y_j t) = \sum_{r=0}^{\infty} e_r(y) t^r$$

for the Schur Q-functions $q_r(x)$ and elementary symmetric polynomials $e_r(y)$. If λ is any partition, let λ' denote the partition conjugate to λ, whose diagram is the transpose of the diagram of λ. Then the Schur S-polynomial $s_{\lambda'}(y)$ may be defined by the equation

$$s_{\lambda'} = \prod_{i<j} (1 - R_{ij})\, e_\lambda.$$

Furthermore, given any strict partition λ of length $\ell(\lambda)$, the Schur Q-function $Q_\lambda(x)$ is defined by the raising operator expression

$$Q_\lambda = \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}}\, q_\lambda$$

and the P-function $P_\lambda(x)$ is given by $P_\lambda = 2^{-\ell(\lambda)}\, Q_\lambda$. In particular $P_0 = 1$ and for each integer $r \geq 1$, we have $P_r = q_r/2$.

Following [BKT2], for each integer r, define $\vartheta_r = \vartheta_r(x\,; y)$ by

$$\vartheta_r = \sum_{i \geq 0} q_{r-i}(x) e_i(y).$$

We let $\Gamma^{(k)} = \mathbb{Z}[\vartheta_1, \vartheta_2, \vartheta_3, \ldots]$ be the ring of theta polynomials. There is a ring isomorphism $\Omega^{(k)} \to \Gamma^{(k)}$ sending ω_r to ϑ_r for all r. Let λ be a k-strict partition, and consider the raising operator

$$\widetilde{R}^\lambda = \prod_{i<j}(1-R_{ij}) \prod_{i<j\,:\,\lambda_i+\lambda_j>2k+j-i}(1+R_{ij})^{-1}.$$

The theta polynomial $\Theta_\lambda(x\,;y)$ is defined by the equation $\Theta_\lambda = \widetilde{R}^\lambda\,\vartheta_\lambda$. The polynomials Θ_λ for all k-strict partitions λ form a $\mathbb{Z}$-basis of $\Gamma^{(k)}$.

Set

$$\eta_r(x\,;y) = \begin{cases} e_r(y) + 2\sum_{i=0}^{r-1} P_{r-i}(x)e_i(y) & \text{if } r<k, \\ \sum_{i=0}^{r} P_{r-i}(x)e_i(y) & \text{if } r \geq k \end{cases}$$

and

$$\eta'_k(x\,;y) = \sum_{i=0}^{k-1} P_{k-i}(x)e_i(y).$$

For any $r \geq 0$, we have

$$\vartheta_r = \begin{cases} \eta_r & \text{if } r<k, \\ \eta_k+\eta'_k & \text{if } r=k, \\ 2\eta_r & \text{if } r>k, \end{cases}$$

while $\eta_k - \eta'_k = e_k(y)$. Following [BKT3], we define the ring of eta polynomials

$$B^{(k)} = \mathbb{Z}[\eta_1,\ldots,\eta_{k-1},\eta_k,\eta'_k,\eta_{k+1}\ldots].$$

3.2. **The reduction formulas for $\widehat{H}_\lambda$ and $\widetilde{H}_\lambda$.** For any k-strict partition λ, define $\widehat{\Theta}_\lambda(x\,;y)$ and $\widehat{H}_\lambda(x\,;y)$ by the equations

$$\widehat{\Theta}_\lambda = R^\lambda\,\vartheta_\lambda \quad \text{and} \quad \widehat{H}_\lambda = 2^{-\ell_k(\lambda)}\,\widehat{\Theta}_\lambda.$$

and let $\tilde{x} = (x_2, x_3, \ldots)$.

Proposition 1. *For any k-strict partition λ, we have the reduction formula*

$$\widehat{H}_\lambda(x\,;y) = \sum_{p=0}^{\infty} x_1^p \sum_{\substack{\mu \rightsquigarrow \lambda \\ |\mu|=|\lambda|-p}} 2^{\widehat{n}(\lambda/\mu)}\,\widehat{H}_\mu(\tilde{x}\,;y). \tag{11}$$

Proof. We compute that

$$\sum_{r=0}^{\infty}\vartheta_r(x\,;y)t^r = \frac{1+x_1t}{1-x_1t}\prod_{i=2}^{\infty}\frac{1+x_it}{1-x_it}\prod_{j=1}^{k}(1+y_jt) = \sum_{i=0}^{\infty}x_1^i\,2^{\#i}\sum_{s=0}^{\infty}\vartheta_s(\tilde{x}\,;y)t^{s+i}$$

and therefore, for any integer sequence μ, we have

$$\vartheta_\mu(x\,;y) = \sum_{\alpha\geq 0} x_1^{|\alpha|}\,2^{\#\alpha}\,\vartheta_{\mu-\alpha}(\tilde{x}\,;y) \tag{12}$$

summed over all compositions α. If R denotes any raising operator, then

$$R\,\vartheta_\mu(x\,;y) = \vartheta_{R\mu}(x\,;y) = \sum_{\alpha\geq 0} x_1^{|\alpha|}\,2^{\#\alpha}\,\vartheta_{R\mu-\alpha}(\tilde{x}\,;y) = \sum_{\alpha\geq 0} x_1^{|\alpha|}\,2^{\#\alpha}\,R\,\vartheta_{\mu-\alpha}(\tilde{x}\,;y).$$

Taking μ equal to a k-strict partition λ and applying the raising operator R^λ to both sides of (12), we therefore obtain

$$\widehat{\Theta}_\lambda(x\,;y) = \sum_{\alpha\geq 0} x_1^{|\alpha|}\,2^{\#\alpha}\,\widehat{\Theta}^{\mathcal{C}(\lambda)}_{\lambda-\alpha}(\tilde{x}\,;y) = \sum_{p=0}^{\infty} x_1^p \sum_{|\alpha|=p} 2^{\#\alpha}\,\widehat{\Theta}^{\mathcal{C}(\lambda)}_{\lambda-\alpha}(\tilde{x}\,;y),$$

where $\widehat{\Theta}^{\mathcal{C}(\lambda)}_{\lambda-\alpha} = R^\lambda\,\vartheta_{\lambda-\alpha}$ by definition. We now use the mirror identity (6) and the equation $\widehat{\Theta}_\lambda = 2^{\ell_k(\lambda)}\,\widehat{H}_\lambda$ to complete the proof. □

For any k-strict partition λ of positive type, we define $\widetilde{H}_\lambda(x\,;y)$ by the equation

$$\widetilde{H}_\lambda = 2^{-\ell_k(\lambda)}\,e_k(y)\,\Theta_{\lambda-k} \tag{13}$$

where $\lambda - k$ means λ with one part equal to k removed. If λ does not have a part equal to k, we agree that $\widetilde{H}_\lambda = 0$. The following analogue of Proposition 1 is valid for the polynomials $\widetilde{H}(x\,;y)$.

Proposition 2. *For any k-strict partition λ of positive type, we have the reduction formula*

$$\widetilde{H}_\lambda(x\,;y) = \sum_{p=0}^{\infty} x_1^p \sum_{\substack{\mu \rightsquigarrow \lambda \\ |\mu|=|\lambda|-p}} 2^{\widehat{n}(\lambda/\mu)}\,\widetilde{H}_\mu(\tilde{x}\,;y) \tag{14}$$

where the sum is over k-strict partitions $\mu \subset \lambda$ of positive type such that λ/μ is a k'-horizontal strip.

Proof. This follows immediately from (13) and the reduction formula for the theta polynomial $\Theta_{\lambda-k}(x\,;y)$. For the latter, see [T1, Theorem 4]. □

3.3. **Typed k-strict partitions and eta polynomials.** A *typed k-strict partition* λ consists of a k-strict partition $(\lambda_1,\ldots,\lambda_\ell)$ together with an integer $\text{type}(\lambda) \in \{0,1,2\}$, such that $\text{type}(\lambda) > 0$ if and only if $\lambda_j = k$ for some index j. The type is usually omitted from the notation for the pair $(\lambda, \text{type}(\lambda))$. Suppose that λ is a typed k-strict partition and R is any finite monomial in the operators R_{ij} which appears in the expansion of the power series R^λ in (1). If $\text{type}(\lambda) = 0$, then set $R \star \vartheta_\lambda = \vartheta_{R\lambda}$. Suppose that $\text{type}(\lambda) > 0$, let m be the smallest index such that $\lambda_m = k$, and set $\widehat{\alpha} = (\alpha_1,\ldots,\alpha_{m-1},\alpha_{m+1},\ldots,\alpha_\ell)$ for any integer sequence α of length ℓ. If R involves any factors R_{ij} with $i = m$ or $j = m$,[1] then let $R\star\vartheta_\lambda = \frac{1}{2}\,\vartheta_{R\lambda}$. If R has no such factors, then let

$$R \star \vartheta_\lambda = \begin{cases} \eta_k\,\vartheta_{\widehat{R\lambda}} & \text{if } \text{type}(\lambda) = 1, \\ \eta'_k\,\vartheta_{\widehat{R\lambda}} & \text{if } \text{type}(\lambda) = 2. \end{cases}$$

The *eta polynomial* $H_\lambda = H_\lambda(x\,;y)$ is the element of $B^{(k)}$ defined by the raising operator formula

$$H_\lambda = 2^{-\ell_k(\lambda)} R^\lambda \star \vartheta_\lambda. \tag{15}$$

We note that, following [BKT3, T2, T3], the term 'eta polynomial' is used to denote both the Giambelli polynomial (15) in the formal variables η_r, η'_k and its image in the ring $B^{(k)}$ (we are only concerned with the latter here). The *type* of the polynomial H_λ is the same as the type of λ. It was shown in [BKT3, Theorem 4] that the H_λ for typed k-strict partitions λ form a $\mathbb{Z}$-basis of the ring $B^{(k)}$.

[1] Note that if $i < m < j$, then the factorization $R_{ij} = R_{im}R_{mj}$ is not allowed.

For any typed k-strict partition λ, we define $\widehat{H}_\lambda$ and $\widetilde{H}_\lambda$ as above, by ignoring the type of λ. It follows that the definition (15) of H_λ is equivalent to the formula

$$H_\lambda = \begin{cases} \widehat{H}_\lambda & \text{if type}(\lambda)=0, \\ \frac{1}{2}(\widehat{H}_\lambda + \widetilde{H}_\lambda) & \text{if type}(\lambda)=1, \\ \frac{1}{2}(\widehat{H}_\lambda - \widetilde{H}_\lambda) & \text{if type}(\lambda)=2. \end{cases} \tag{16}$$

Lemma 3. *Let λ be a typed k-strict partition and $H_\lambda(0\,;y)$ be obtained from $H_\lambda(x\,;y)$ by substituting $x_i=0$ for all $i\geq 1$. Then we have*

$$H_\lambda(0\,;y) = \begin{cases} 0 & \text{if } \lambda_1 > k, \\ s_{\lambda'}(y) & \text{if } \lambda_1 = k \text{ and type}(\lambda)=1, \\ 0 & \text{if } \lambda_1 = k \text{ and type}(\lambda)=2, \\ s_{\lambda'}(y) & \text{if } \lambda_1 < k \end{cases}$$

where λ' denotes the partition conjugate to λ.

Proof. The raising operator definition of $\widehat{\Theta}_\lambda$ gives

$$\widehat{\Theta}_\lambda(0\,;y) = R^\lambda\, e_\lambda(y) \tag{17}$$

where $e_\lambda = \prod_i e_{\lambda_i}(y)$ and $e_r(y)$ denotes the r-th elementary symmetric polynomial in y. Since $e_r(y)=0$ for $r>k$, we deduce from (17) that $\widehat{H}_\lambda(0\,;y)=0$ unless $\lambda_1 \leq k$. In the latter case we have $\ell_k(\lambda)=0$, $\mathcal{C}(\lambda)=\emptyset$, and

$$\widehat{\Theta}_\lambda(0\,;y) = \Theta_\lambda(0\,;y) = \prod_{i<j}(1-R_{ij})\,e_\lambda(y) = s_{\lambda'}(y). \tag{18}$$

Suppose that $\lambda_1 = k$, so that type$(\lambda)>0$. If type$(\lambda)=1$, then equations (13), (16) and (18) give

$$\begin{aligned} H_\lambda(0\,;y) &= \frac{1}{2}\left(\widehat{\Theta}_\lambda(0\,;y) + e_k(y)\Theta_{\lambda-k}(0\,;y)\right) \\ &= \frac{1}{2}\left(s_{\lambda'}(y) + e_k(y)s_{(\lambda-k)'}(y)\right) = s_{\lambda'}(y), \end{aligned}$$

since the classical type A Pieri rule gives $e_k(y)s_{(\lambda-k)'}(y) = s_{\lambda'}(y)$. We similarly compute that $H_\lambda(0\,;y)=0$ if type$(\lambda)=2$. Finally, if $\lambda_1<k$ then type$(\lambda)=0$ and $H_\lambda(0\,;y) = \widehat{\Theta}_\lambda(0\,;y) = s_{\lambda'}(y)$, using (18) again. □

3.4. **Tableau formulas.** In this subsection we will obtain a description of the eta polynomial $H_\lambda(x\,;y)$ as a sum over tableaux which are fillings of the Young diagram of λ. We first prove a reduction formula for the x variables which appear in H_λ.

Definition 2. If λ and μ are typed k-strict partitions with $\mu\subset\lambda$, we write $\mu \rightsquigarrow \lambda$ and say that λ/μ is a *typed k'-horizontal strip* if the underlying k-strict partitions are such that λ/μ is a k'-horizontal strip and in addition type$(\lambda)+$type$(\mu)\neq 3$. In this case we set $n(\lambda/\mu) = N(\mathbb{A})-1$, where the set $\mathbb{A}$ and integer $N(\mathbb{A})$ are defined as in §2.6.

Theorem 2. *For any typed k-strict partition λ, we have the reduction formula*

$$H_\lambda(x\,;y) = \sum_{p=0}^{\infty} x_1^p \sum_{\substack{\mu\rightsquigarrow\lambda \\ |\mu|=|\lambda|-p}} 2^{n(\lambda/\mu)}\, H_\mu(\tilde{x}\,;y) \tag{19}$$

where $\tilde{x} = (x_2, x_3, \ldots)$ *and the inner sum is over typed* k*-strict partitions* μ *with* $\mu \rightsquigarrow \lambda$ *and* $|\mu| = |\lambda| - p$.

Proof. Assume that $\mathrm{type}(\lambda) = 1$; the proof in the other cases is similar. Using Definition 15 and equations (11) and (14) we obtain

$$H_\lambda(x\,;y) = \sum_{p=0}^{\infty} x_1^p \sum_{\substack{\mu \rightsquigarrow \lambda \\ |\mu| = |\lambda| - p}} 2^{\widehat{n}(\lambda/\mu)-1} \, (\widehat{H}_\mu(\tilde{x}\,;y) + \widetilde{H}_\mu(\tilde{x}\,;y)) \tag{20}$$

where the inner sum is over all k-strict partitions $\mu \subset \lambda$ with $|\mu| = |\lambda| - p$ such that λ/μ is a k'-horizontal strip. The result follows by combining equation (20) with (10) and (16). □

Let $\mathbf{P}$ denote the ordered alphabet $\{\widehat{1} < \widehat{2} < \cdots < \widehat{k} < 1, 1^\circ < 2, 2^\circ < \cdots\}$. The symbols $\widehat{1}, \ldots, \widehat{k}$ are said to be *marked*, while the rest are *unmarked*. Suppose that λ is any typed k-strict partition.

Definition 3. a) A *typed* k'*-tableau* T of shape λ/μ is a sequence of typed k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r = \lambda \tag{21}$$

such that λ^i/λ^{i-1} is a typed k'-horizontal strip for $1 \leq i \leq r$. We represent T by a filling of the boxes in λ/μ with unmarked elements of $\mathbf{P}$ which is weakly increasing along each row and down each column, such that for each i, the boxes in T with entry i or i° form the skew diagram λ^i/λ^{i-1}, and we use i (resp. i°) if and only if $\mathrm{type}(\lambda^i) \neq 2$ (resp. $\mathrm{type}(\lambda^i) = 2$), for every $i \geq 1$. A *standard typed* k'*-tableau* on λ/μ is a typed k'-tableau T of shape λ/μ such that the entries $1, 2, \ldots, |\lambda - \mu|$, circled or not, each appear exactly once in T. For any typed k'-tableau T we define

$$n(T) = \sum_i n(\lambda^i/\lambda^{i-1}) \quad \text{and} \quad x^T = \prod_i x_i^{m_i}$$

where m_i denotes the number of times that i or i° appears in T.

b) A *typed* k'*-bitableau* U of shape λ is a filling of the boxes in λ with elements of $\mathbf{P}$ which is weakly increasing along each row and down each column, such that (i) the unmarked entries form a typed k'-tableau T of shape λ/μ with $\mathrm{type}(\mu) \neq 2$, and (ii) the marked entries are a filling of μ which is strictly increasing along each row. We define

$$n(U) = n(T) \quad \text{and} \quad (xy)^U = x^T \prod_{j=1}^{k} y_j^{n_j}$$

where n_j denotes the number of times that $\widehat{j}$ appears in U.

Theorem 3. *For any typed* k*-strict partition* λ*, we have*

$$H_\lambda(x\,;y) = \sum_U 2^{n(U)} (xy)^U$$

where the sum is over all typed k'*-bitableaux* U *of shape* λ.

Proof. Let m be a positive integer, $x^{(m)} = (x_1, \ldots, x_m)$, and let $H_\lambda(x^{(m)}\,;y)$ be the result of substituting $x_i = 0$ for $i > m$ in $H_\lambda(x\,;y)$. It follows from equation (19) that

$$H_\lambda(x^{(m)}\,;y) = \sum_{p=0}^{\infty} x_m^p \sum_{\substack{\mu \rightsquigarrow \lambda \\ |\mu|=|\lambda|-p}} 2^{n(\lambda/\mu)}\, H_\mu(x^{(m-1)}\,;y). \tag{22}$$

Iterating equation (22) m times produces

$$H_\lambda(x^{(m)}\,;y) = \sum_{\mu,T} 2^{n(T)}\, x^T H_\mu(0\,;y)$$

where the sum is over all typed k-strict partitions $\mu \subset \lambda$ and typed k-tableau T of shape λ/μ with no entries greater than m. We deduce from Lemma 3 that $H_\mu(0\,;y) = 0$ unless $\mu_1 \leq k$ and $\mathrm{type}(\mu) \neq 2$, in which case $H_\mu(0\,;y) = s_{\mu'}(y)$. The combinatorial definition of Schur S-functions [M, I.(5.12)] states that

$$s_{\mu'}(y) = \sum_S y^S$$

summed over all semistandard Young tableaux S of shape μ' with entries from 1 to k. We conclude that

$$H_\lambda(x^{(m)}\,;y) = \sum_U 2^{n(U)}\, (xy)^U$$

summed over all typed k'-bitableaux U of shape λ with no entries greater than m. The theorem follows by letting m tend to infinity. □

Example 2. Let $k = 1$, $\lambda = (3,1)$ with $\mathrm{type}(\lambda) = 1$, and consider the alphabet $\mathbf{P}_{1,2} = \{\widehat{1} < 1, 1^\circ < 2, 2^\circ\}$. There are thirteen typed $1'$-bitableaux U of shape λ with entries in $\mathbf{P}_{1,2}$. The three typed $1'$-bitableaux $\begin{matrix}\widehat{1}\,1\,2\\ \widehat{1}\end{matrix}$, $\begin{matrix}\widehat{1}\,2\,2\\ 1\end{matrix}$, and $\begin{matrix}\widehat{1}\,1\,2\\ 1\end{matrix}$ satisfy $n(U) = 1$, while the ten typed $1'$-bitableaux

$$\begin{matrix}1\,1\,1 & 1\,1\,2 & 1\,2\,2 & 1\,2\,2 & \widehat{1}\,1\,1\\ 2 & 2 & 1 & 2 & 1\end{matrix}$$

$$\begin{matrix}\widehat{1}\,1\,1 & \widehat{1}\,1\,2 & \widehat{1}\,2\,2 & \widehat{1}\,1\,1 & \widehat{1}\,2\,2\\ 2 & 2 & 2 & \widehat{1} & \widehat{1}\end{matrix}$$

satisfy $n(U) = 0$. If $H_{3,1}$ denotes the eta polynomial indexed by λ, then we deduce from Theorem 3 that

$$\begin{aligned}H_{3,1}(x_1,x_2\,;y_1) &= (x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3) + (x_1^3 + 3x_1^2x_2 + 3x_1x_2^2 + x_2^3)\,y_1\\ &\quad + (x_1^2 + 2x_1x_2 + x_2^2)\,y_1^2\\ &= P_{3,1}(x_1,x_2) + (P_3(x_1,x_2) + P_{2,1}(x_1,x_2))\;y_1 + P_2(x_1,x_2)\,y_1^2.\end{aligned}$$

We similarly find that for $\lambda = (3,1)$ of type 2, there are six typed $1'$-bitableaux U of shape λ with entries in $\mathbf{P}_{1,2}$, namely

$$\begin{matrix}1\,1\,1 & 1\,1\,2^\circ & 1^\circ\,2^\circ\,2^\circ & 1^\circ\,2^\circ\,2^\circ & \widehat{1}\,1\,1 & \widehat{1}\,1\,2^\circ\\ 2^\circ & 2^\circ & 1^\circ & 2^\circ & 2^\circ & 2^\circ\end{matrix}$$

all of which satisfy $n(U)=0$. If $H'_{3,1}$ denotes the eta polynomial indexed by λ, we deduce from Theorem 3 that

$$\begin{aligned} H'_{3,1}(x_1,x_2\,;y_1) &= (x_1^3x_2+2x_1^2x_2^2+x_1x_2^3)+(x_1^2x_2+x_1x_2^2)\,y_1 \\ &= P_{3,1}(x_1,x_2)+P_{2,1}(x_1,x_2)\,y_1. \end{aligned}$$

Notice that $H_{3,1}+H'_{3,1}=\frac{1}{2}\Theta_{3,1}$, and compare with [T1, Example 7].

Theorem 3 motivates the following definition.

Definition 4. For λ and μ any two typed k-strict partitions with $\mu\subset\lambda$, let

$$E_{\lambda/\mu}(x)=\sum_T 2^{n(T)}\,x^T$$

where the sum is over all typed k'-tableaux T of shape λ/μ.

Example 3. (a) There is an involution j on the set of all typed k-strict partitions, which is the identity on partitions of type 0 and exchanges type 1 and type 2 partitions of the same shape. If T is a typed k'-tableau of shape λ/μ then we let $j(T)$ be the typed k'-tableau of shape $j(\lambda)/j(\mu)$ obtained by applying j to each typed partition λ^i which appears in the sequence (21) determined by T. Then j is an involution on the set of all typed k'-tableaux and it follows from Definition 4 that

$$E_{\lambda/\mu}(x)=E_{j(\lambda)/j(\mu)}(x).$$

(b) Suppose that $\mu_i\geq\min(k,\lambda_i)$ for all i. Then Definition 4 becomes the tableau based definition of skew Schur P-functions found e.g. in [M, III.(8.16)]. We deduce that

$$E_{\lambda/\mu}(x)=P_{\lambda/\mu}(x).$$

(c) Suppose that $\lambda_1<k$, so in particular $\mathcal{C}(\lambda)=\emptyset$. Then $n(\lambda/\mu)$ is equal to the number of edge-connected components of λ/μ, for any partition $\mu\subset\lambda$. It follows from Worley [W, §2.7] that

$$\sum_T 2^{n(T)}\,x^T=S_{\lambda/\mu}(x):=\det(q_{\lambda_i-\mu_j+j-i}(x))_{i,j}.$$

We therefore have $E_{\lambda/\mu}(x)=S_{\lambda/\mu}(x)$ and

$$H_\lambda(x\,;y)=\sum_{\mu\subset\lambda}S_{\lambda/\mu}(x)\,s_{\mu'}(y),$$

in agreement with [BKT3, Prop. 5.4(a)].

(d) Suppose that there is only one variable x. Then we have

$$E_{\lambda/\mu}(x)=\begin{cases}2^{n(\lambda/\mu)}\,x^{|\lambda-\mu|} & \text{if } \lambda/\mu \text{ is a typed } k'\text{-horizontal strip,}\\ 0 & \text{otherwise.}\end{cases}$$

Corollary 1. (a) *Let $x=(x_1,x_2,\ldots)$ and $x'=(x'_1,x'_2,\ldots)$ be two sets of variables, and let λ be any typed k-strict partition. Then we have*

$$H_\lambda(x,x'\,;y)=\sum_{\mu\subset\lambda}E_{\lambda/\mu}(x)\,H_\mu(x'\,;y)\,, \tag{23}$$

$$H_\lambda(x\,;y)=\sum_{\mu\subset\lambda}E_{\lambda/\mu}(x)\,s_{\mu'}(y)\,, \tag{24}$$

and

$$E_\lambda(x,x') = \sum_{\mu\subset\lambda} E_{\lambda/\mu}(x)\,E_\mu(x'), \tag{25}$$

where the sums are over all typed k-strict partitions $\mu \subset \lambda$.

(b) *For any two typed k-strict partitions λ and μ with $\mu \subset \lambda$, we have*

$$E_{\lambda/\mu}(x,x') = \sum_{\nu} E_{\lambda/\nu}(x)\,E_{\nu/\mu}(x'), \tag{26}$$

where the sum is over all typed k-strict partitions ν with $\mu \subset \nu \subset \lambda$.

Proof. The proof is the same as that for Corollaries 6 and 7 in [T1]. □

Since $H_\lambda(x,x'\,;y)$ is symmetric in the variables (x,x') and the $H_\mu(x'\,;y)$ are linearly independent, it follows immediately from identity (23) that $E_{\lambda/\mu}(x)$ is a *symmetric function* in the variables x. These functions will be studied further in the next section.

4. Stanley symmetric functions and skew elements

4.1. **Type D Stanley functions and Schubert polynomials.** Let $\widetilde{W}_{n+1}$ be the Weyl group for the root system of type D_{n+1}, and set $\widetilde{W}_\infty = \bigcup_n \widetilde{W}_{n+1}$. The elements of $\widetilde{W}_{n+1}$ may be represented as signed permutations of the set $\{1,\ldots,n+1\}$; we will denote a sign change by a bar over the corresponding entry. The group $\widetilde{W}_{n+1}$ is generated by the simple transpositions $s_i = (i,i+1)$ for $1 \leq i \leq n$, and an element s_0 which acts on the right by

$$(u_1,u_2,\ldots,u_{n+1})s_0 = (\overline{u}_2,\overline{u}_1,u_3,\ldots,u_{n+1}).$$

Every element $w \in \widetilde{W}_\infty$ can be expressed as a product of $w = s_{a_1}s_{a_2}\cdots s_{a_r}$ of simple reflections s_i. If the length r of such an expression is minimal then the sequence of indices $(a_1,\ldots,a_r)$ is called a *reduced word* for w, and r is called the *length* of w, denoted $\ell(w)$. A factorization $w = uv$ in $\widetilde{W}_\infty$ is *reduced* if $\ell(w) = \ell(u)+\ell(v)$.

Following [FS, FK1, FK2, L], we will use the nilCoxeter algebra $\mathcal{W}_{n+1}$ of $\widetilde{W}_{n+1}$ to define type D Stanley symmetric functions and Schubert polynomials. $\mathcal{W}_{n+1}$ is the free associative algebra with unity generated by the elements $u_0,u_1,\ldots,u_n$ modulo the relations

$$\begin{array}{rcll}
u_i^2 & = & 0 & i \geq 0\ ; \\
u_0u_1 & = & u_1u_0 & \\
u_0u_2u_0 & = & u_2u_0u_2 & \\
u_iu_{i+1}u_i & = & u_{i+1}u_iu_{i+1} & i > 0\ ; \\
u_iu_j & = & u_ju_i & j > i+1, \text{ and } (i,j) \neq (0,2).
\end{array}$$

For any $w \in \widetilde{W}_{n+1}$, choose a reduced word $a_1\cdots a_\ell$ for w and define $u_w = u_{a_1}\ldots u_{a_\ell}$. Since the last four relations listed are the Coxeter relations for D_{n+1}, it is clear that u_w is well defined, and that the u_w for $w \in \widetilde{W}_{n+1}$ form a free $\mathbb{Z}$-basis of $\mathcal{W}_{n+1}$.

Let t be an indeterminate and define

$$A_i(t) = (1+tu_n)(1+tu_{n-1})\cdots(1+tu_i)\ ;$$
$$D(t) = (1+tu_n)\cdots(1+tu_2)(1+tu_1)(1+tu_0)(1+tu_2)\cdots(1+tu_n).$$

According to [L, Lemma 4.24], for any commuting variables s,t we have $D(s)D(t) = D(t)D(s)$. Set $x = (x_1, x_2, \ldots)$ and consider the product $D(x) := D(x_1)D(x_2)\cdots$. We deduce that the functions $E_w(x)$ in the formal power series expansion

$$D(x) = \sum_{w\in \widetilde{W}_{n+1}} E_w(x)\, u_w \tag{27}$$

are symmetric functions in x. The E_w are the *type D Stanley symmetric functions*, introduced and studied in [BH, L].

Let $y = (y_1, y_2, \ldots)$. The Billey-Haiman *type D Schubert polynomials* $\mathfrak{D}_w(x\,;y)$ for $w \in \widetilde{W}_{n+1}$ are defined by expanding the formal product

$$D(x)A_1(y_1)A_2(y_2)\cdots A_n(y_n) = \sum_{w\in \widetilde{W}_{n+1}} \mathfrak{D}_w(x\,;y)\, u_w. \tag{28}$$

The above definition is equivalent to the one in [BH]. Observe that $\mathfrak{D}_w(x\,;y)$ is a polynomial in the y variables, whose coefficients are formal power series in the x variables. One checks that $\mathfrak{D}_w$ is stable under the natural inclusion of $\widetilde{W}_n$ in $\widetilde{W}_{n+1}$, and hence well defined for $w \in \widetilde{W}_\infty$. We also deduce the following result from (27) and (28).

Proposition 3. *Let $w \in \widetilde{W}_\infty$ and $x' = (x'_1, x'_2, \ldots)$. Then we have*

$$E_w(x,x') = \sum_{uv=w} E_u(x)E_v(x') \tag{29}$$

and

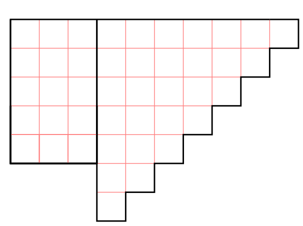
$$\mathfrak{D}_w(x,x'\,;y) = \sum_{uv=w} E_u(x)\, \mathfrak{D}_v(x'\,;y) \tag{30}$$

where the sums are over all reduced factorizations $uv = w$ in $\widetilde{W}_\infty$.

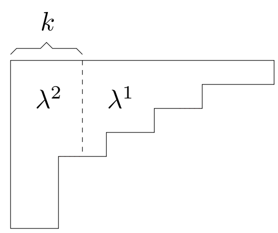
4.2. **The Grassmannian elements of $\widetilde{W}_\infty$.** For $k \neq 1$, an element $w \in \widetilde{W}_\infty$ is k-Grassmannian if $\ell(ws_i) = \ell(w)+1$ for all $i \neq k$. We say that w is 1-Grassmannian if $\ell(ws_i) = \ell(w)+1$ for all $i \geq 2$. The elements of $\widetilde{W}_{n+1}$ index the Schubert classes in the cohomology ring of the flag variety SO_{2n+2}/B, which contains $\mathrm{H}^*(\mathrm{OG}(n+1-k, 2n+2), \mathbb{Z})$ as the subring spanned by Schubert classes given by k-Grassmannian elements. If $\widetilde{\mathcal{P}}(k,n)$ denotes the set of typed k-strict partitions whose diagrams fit inside a rectangle of size $(n+1-k)\times(n+k)$, then each element λ in $\widetilde{\mathcal{P}}(k,n)$ corresponds to a k-Grassmannian element $w_\lambda \in \widetilde{W}_{n+1}$ which we proceed to describe.

The typed k-strict partitions λ in $\widetilde{\mathcal{P}}(k,n)$ are those whose Young diagram fits inside the non-convex polygon Π obtained by attaching an $(n+1-k)\times k$ rectangle to the left side of a staircase partition with n rows. When $n=7$ and $k=3$, the polygon Π looks as follows.

Π =



The boxes of the staircase partition that are outside λ lie in south-west to north-east diagonals. Such a diagonal is called *related* if it is k'-related to one of the bottom boxes in the first k columns of λ, or to any box $[0, i]$ for which $\lambda_1 < i \leq k$; the remaining diagonals are called *non-related.* Let $u_1 < u_2 < \cdots < u_k$ denote the lengths of the related diagonals. Moreover, let λ^1 be the strict partition obtained by removing the first k columns of λ, let λ^2 be the partition of boxes contained in the first k columns of λ, and set $r = \ell(\lambda^1) = \ell_k(\lambda)$.



If $\mathrm{type}(\lambda) = 1$, then the k-Grassmannian element corresponding to λ is given by

$$w_\lambda = (u_1 + 1, \ldots, u_k + 1, \overline{(\lambda^1)_1 + 1}, \ldots, \overline{(\lambda^1)_r + 1}, \widetilde{1}, v_1 + 1, \ldots, v_{n-k-r} + 1),$$

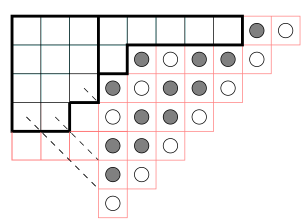
while if $\mathrm{type}(\lambda) = 2$, then

$$w_\lambda = (\overline{u_1 + 1}, \ldots, u_k + 1, \overline{(\lambda^1)_1 + 1}, \ldots, \overline{(\lambda^1)_r + 1}, \widetilde{1}, v_1 + 1, \ldots, v_{n-k-r} + 1),$$

where $v_1 < \cdots < v_{n-k-r}$ are the lengths of the non-related diagonals. Here $\widetilde{1}$ is equal to 1 or $\overline{1}$, so that the total number of barred entries in w_λ is even. Finally, if $\mathrm{type}(\lambda) = 0$, then $u_1 = 0$, that is, one of the related diagonals has length zero. In this case

$$w_\lambda = (\widetilde{1}, u_2 + 1, \ldots, u_k + 1, \overline{(\lambda^1)_1 + 1}, \ldots, \overline{(\lambda^1)_r + 1}, v_1 + 1, \ldots, v_{n+1-k-r} + 1),$$

where $v_1 < \cdots < v_{n+1-k-r}$ are the lengths of the non-related diagonals. For example, the element $\lambda = (8, 4, 3, 2) \in \widetilde{\mathcal{P}}(3, 7)$ of type 1 corresponds to $w_\lambda = (3, 5, 7, \overline{6}, \overline{2}, 1, 4, 8)$.

$\lambda =$ 

The element $w_\lambda \in \widetilde{W}_\infty$ depends on λ and k, but is independent of n. Furthermore, it was proved in [BKT3, Prop. 6.3] that

$$H_\lambda(x\,; y) = \mathfrak{D}_{w_\lambda}(x\,; y) \tag{31}$$

for any typed k-strict partition λ. The main point of (31) is the expression of $\mathfrak{D}_{w_\lambda}$ via raising operators, as in (15). Observe also that the equality (31) is taking place in the full ring $B^{(0)}[z] = \mathbb{Z}[P_1(x), P_2(x), \ldots; z_1, z_2, \ldots]$ of type D Billey-Haiman Schubert polynomials, where there are relations among the generators $P_r(x)$. Moreover, these relations are used crucially in its proof, which is given in [BKT3].

4.3. **The skew elements of $\widetilde{W}_\infty$.** The following definition can be formulated for any Coxeter group.

Definition 5. An element $w \in \widetilde{W}_\infty$ is called *skew* if there exists a typed k-strict partition λ (for some k) and a reduced factorization $w_\lambda = ww'$ in $\widetilde{W}_\infty$.

Note that if we have a reduced factorization $w_\lambda = ww'$ in $\widetilde{W}_{n+1}$ for some typed k-strict partition $\lambda \in \widetilde{\mathcal{P}}(k,n)$, then the right factor w' is k-Grassmannian, and therefore equal to w_μ for some typed k-strict partition $\mu \in \widetilde{\mathcal{P}}(k,n)$.

Proposition 4. *Suppose that w is a skew element of $\widetilde{W}_\infty$, and let λ and μ be typed k-strict partitions such that the factorization $w_\lambda = ww_\mu$ is reduced. Then we have $\mu \subset \lambda$ and $E_w(x) = E_{\lambda/\mu}(x)$.*

Proof. By combining (31) with (23) and (30), we see that

$$\sum_{\mu\subset\lambda} E_{\lambda/\mu}(x)\, H_\mu(x'\,;y) = H_\lambda(x,x'\,;y) = \sum_{uv=w_\lambda} E_u(x)\,\mathfrak{D}_v(x'\,;y) \tag{32}$$

where the first sum is over all typed k-strict partitions $\mu \subset \lambda$ and the second sum is over all reduced factorizations $uv = w_\lambda$. The right factor v in any such reduced factorization must equal w_ν for some typed k-strict partition ν, and therefore $\mathfrak{D}_v(x'\,;y) = H_\nu(x'\,;y)$. Since the $H_\nu(x'\,;y)$ for ν a typed k-strict partition form a $\mathbb{Z}$-basis for the ring $B^{(k)}(x'\,;y)$ of eta polynomials in x' and y, the desired result follows. □

Definition 6. Let λ and μ be typed k-strict partitions in $\widetilde{\mathcal{P}}(k,n)$ with $\mu \subset \lambda$. We say that (λ,μ) is a *compatible pair* if there is a reduced word for w_λ whose last $|\mu|$ entries form a reduced word for w_μ; equivalently, if we have $\ell(w_\lambda w_\mu^{-1}) = |\lambda - \mu|$.

From Proposition 4 we immediately deduce the next result.

Corollary 2. *Let (λ,μ) be a compatible pair of typed k-strict partitions. Then there is a 1-1 correspondence between reduced factorizations of $w_\lambda w_\mu^{-1}$ and typed k-strict partitions ν with $\mu \subset \nu \subset \lambda$ such that (λ,ν) and (ν,μ) are compatible pairs.*

Theorem 4. *Let λ and μ be typed k-strict partitions in $\widetilde{\mathcal{P}}(k,n)$ with $\mu \subset \lambda$. Then the following conditions are equivalent:* (a) *$E_{\lambda/\mu}(x) \neq 0$;* (b) *$(\lambda,\mu)$ is a compatible pair;* (c) *there exists a standard typed k'-tableau on λ/μ. If any of these conditions holds, then $E_{\lambda/\mu}(x) = E_{w_\lambda w_\mu^{-1}}(x)$.*

Proof. Equation (32) may be rewritten in the form

$$\sum_{\mu\subset\lambda} E_{\lambda/\mu}(x)\, H_\mu(x'\,;y) = \sum_{\mu} E_{w_\lambda w_\mu^{-1}}(x)\, H_\mu(x'\,;y) \tag{33}$$

where the second sum is over all $\mu \subset \lambda$ such that (λ,μ) is a compatible pair. It follows that

$$E_{\lambda/\mu}(x) = \begin{cases} E_{w_\lambda w_\mu^{-1}}(x) & \text{if } (\lambda,\mu) \text{ is a compatible pair,} \\ 0 & \text{otherwise.} \end{cases}$$

Since clearly $E_w(x) \neq 0$ for any $w \in \widetilde{W}_\infty$, we deduce that (a) and (b) are equivalent. Suppose now that (λ,μ) is a compatible pair with $|\lambda| = |\mu| + 1$, so that $w_\lambda = s_i w_\mu$ for some $i \geq 0$. Observe that if x is a single variable, then $E_{s_i}(x) = x$, if $i \leq 1$, and

$E_{s_i}(x) = 2x$, if $i > 1$. Therefore $E_{\lambda/\mu}(x) \neq 0$, and we deduce from Example 3(d) that λ/μ must be a typed k'-horizontal strip. Using Corollary 2, it follows that there is a 1-1 correspondence between reduced words for $w_\lambda w_\mu^{-1}$ and sequences of typed k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r = \lambda$$

such that $|\lambda^i| = |\lambda^{i-1}| + 1$ and λ^i/λ^{i-1} is a typed k'-horizontal strip for $1 \leq i \leq r = |\lambda - \mu|$. The latter objects are exactly the standard typed k'-tableaux on λ/μ. This shows that (b) implies (c), and the converse is also clear. □

The previous results show that the non-zero terms in equations (23) and (25) correspond exactly to the terms in equations (30) and (29), respectively, when $w = w_\lambda$ is a k-Grassmannian element of $\widetilde{W}_\infty$.

Corollary 3. *Let $w \in \widetilde{W}_\infty$ be a skew element and (λ, μ) be a compatible pair such that $w_\lambda = ww_\mu$. Then the number of reduced words for w is equal to the number of standard typed k'-tableaux on λ/μ.*

Let ι denote the involution on the set of reduced words which interchanges the letters 0 and 1, for example $\iota(02120) = 12021$. Then the restriction of ι to the set of reduced words representing skew elements of $\widetilde{W}_\infty$ corresponds to the involution j on the set of typed k'-tableaux defined in Example 3(a).

Example 4. Let λ be a typed k-strict partition with type$(\lambda) \in \{0, 1\}$ and let λ^1 and λ^2 be defined as in §4.2. We can form a standard typed k'-tableau on λ by filling the boxes of λ^2, going down the columns from left to right, and then filling the boxes of λ^1, going across the rows from top to bottom. When $k = 2$ and $\lambda = (7, 6, 5, 2)$ with type$(\lambda) = 1$, the typed $2'$-tableau on λ which results is

1	5	9	10	11	12	13
2	6	14	15	16	17	
3	7	18	19	20		
4	8					

which corresponds to the reduced word

$$3\,2\,0\,4\,3\,2\,1\,5\,4\,3\,2\,0\,4\,3\,2\,1\,5\,4\,3\,2$$

for the 2-Grassmannian element $w_\lambda = 23\overline{6541} \in \widetilde{W}_6$. We can form a reduced word for the element $w'_\lambda = \overline{2}3\overline{6541}$ associated to $\lambda = (7, 6, 5, 2)$ with type$(\lambda) = 2$ by applying the involution ι to obtain

$$3\,2\,1\,4\,3\,2\,0\,5\,4\,3\,2\,1\,4\,3\,2\,0\,5\,4\,3\,2.$$

This last word corresponds to the standard typed $2'$-tableau

1	5°	9°	10°	11°	12°	13°
2	6°	14°	15°	16°	17°	
3	7°	18°	19°	20°		
4	8°					

.

Example 5. Consider the 1-Grassmannian element $w = 3\overline{42}1$ in $\widetilde{W}_4$ associated to the typed 1-strict partition $\lambda = (4, 2, 1)$ of type 1. The following table lists the nine reduced words for w and the corresponding standard typed $1'$-tableaux of shape λ.

word	tableau	word	tableau	word	tableau
1320321	1 4 5 6 2 7 3	1323021	1 3 5 6 2 7 4	1232021	1 3 4 5 2 7 6
1230201	1 2 3 5 4 7 6	1230210	1° 2 3 5 4 7 6	1203201	1 2 3 4 5 7 6
1203210	1° 2 3 4 5 7 6	3120321	1 4 5 7 2 6 3	3123021	1 3 5 7 2 6 4

Corollary 4. *For any two typed k-strict partitions λ, μ with $\mu \subset \lambda$, the function $E_{\lambda/\mu}(x)$ is a nonnegative integer linear combination of Schur P-functions.*

Proof. According to [BH] and [L], the type D Stanley symmetric function $E_w(x)$ is a nonnegative integer linear combination of Schur P-functions. The result follows from this fact together with Theorem 4. □

As noted in the introduction, the skew elements of the symmetric group may be identified with the 321-avoiding permutations, following [BJS, S1]. It would be interesting to determine whether the skew elements in $\widetilde{W}_\infty$ (and in the hyperoctahedral group) can also be characterized by pattern avoidance conditions.

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