QUANTUM COHOMOLOGY OF ORTHOGONAL GRASSMANNIANS

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Abstract. Let V be a vector space with a nondegenerate symmetric form and OG be the orthogonal Grassmannian which parametrizes maximal isotropic subspaces in V. We give a presentation for the (small) quantum cohomology ring $QH^*(OG)$ and show that its product structure is determined by the ring of $\widetilde{P}$-polynomials. A 'quantum Schubert calculus' is formulated, which includes quantum Pieri and Giambelli formulas, as well as algorithms for computing Gromov–Witten invariants. As an application, we show that the table of 3-point, genus zero Gromov–Witten invariants for OG coincides with that for a corresponding Lagrangian Grassmannian LG, up to an involution.

1. Introduction

Consider a complex vector space V together with a nondegenerate symmetric form. Our aim is to study the structure of the small quantum cohomology ring of the orthogonal Grassmannian of maximal isotropic subspaces in V. In a companion paper to this one [KT2], we provide a similar analysis in type C, i.e., for the Lagrangian Grassmannian, and the reader is referred there and to [FP] [LT] for further background material. The story in the orthogonal case is similar, but with significant differences, both in the results and in their proofs.

Assuming the dimension of V is even and equals $2n+2$ for some natural number n, then the space of maximal isotropic subspaces of V has two connected components, each isomorphic to the *even orthogonal Grassmannian* or *spinor variety* $OG = OG(n+1, 2n+2) = SO_{2n+2}/P_{n+1}$. Here P_{n+1} is the maximal parabolic subroup of SO_{2n+2} associated to a 'right end root' in the Dynkin diagram of type D_{n+1}. We note that $OG(n+1, 2n+2)$ is isomorphic (in fact projectively equivalent) to the odd orthogonal Grassmannian $OG(n, 2n+1) = SO_{2n+1}/P_n$. Therefore, it suffices to only work with the even orthogonal example and we do so throughout this paper. We agree that a class α in the cohomology $H^{2k}(\mathfrak{X}, \mathbb{Z})$ of a complex variety $\mathfrak{X}$ has degree k to avoid doubling all degrees.

The cohomology ring $H^*(OG, \mathbb{Z})$ has a $\mathbb{Z}$-basis of Schubert classes τ_λ, one for each strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$ with $\lambda_1 \leqslant n$. Their multiplication can be described using the $\widetilde{P}$*-polynomials* of Pragacz and Ratajski [PR]. Let $X = (x_1, \ldots, x_n)$ be an n-tuple of variables and define $\widetilde{P}_0(X) = 1$ and $\widetilde{P}_i(X) = e_i(X)/2$ for each $i > 0$, where $e_i(X)$ denotes the i-th elementary symmetric polynomial in

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X. For nonnegative integers i,j with $i \geqslant j$, set

$$\widetilde{P}_{i,j}(X) = \widetilde{P}_i(X)\widetilde{P}_j(X) + 2\sum_{k=1}^{j-1}(-1)^k \widetilde{P}_{i+k}(X)\widetilde{P}_{j-k}(X) + (-1)^j \widetilde{P}_{i+j}(X), \tag{1}$$

and for any partition λ of length $\ell = \ell(\lambda)$, not necessarily strict, define

$$\widetilde{P}_\lambda(X) = \text{Pfaffian}[\widetilde{P}_{\lambda_i,\lambda_j}(X)]_{1\leqslant i<j\leqslant r}, \tag{2}$$

where $r = 2\lfloor(\ell+1)/2\rfloor$. Let $\mathcal{D}_n$ be the set of strict partitions λ with $\lambda_1 \leqslant n$.

Let Λ'_n denote the $\mathbb{Z}$-algebra generated by the polynomials $\widetilde{P}_\lambda(X)$ for all $\lambda \in \mathcal{D}_n$; Λ'_n is isomorphic to the ring $\mathbb{Z}[X]^{S_n}$ of symmetric polynomials in X. By results of [P, Sect. 6] and [PR] we have that the map sending $\widetilde{P}_\lambda(X)$ to τ_λ for all $\lambda \in \mathcal{D}_n$ extends to a surjective *ring* homomorphism $\phi : \Lambda'_n \to H^*(OG,\mathbb{Z})$ with kernel generated by the relations $\widetilde{P}_{i,i}(X) = 0$ for $1 \leqslant i \leqslant n$. The map ϕ can be realized as evaluation on the Chern roots of the tautological quotient vector bundle Q over OG (note that the top Chern class of Q vanishes). In this way we obtain a presentation for the cohomology ring of OG, and equations (1) and (2) become Giambelli-type formulas, which express the Schubert classes in terms of the special ones.

We present an extension of these results to the (small) quantum cohomology ring of OG, denoted $QH^*(OG)$. This is an algebra over $\mathbb{Z}[q]$, where q is a formal variable of degree $2n$ (the classical formulas are recovered by setting $q = 0$).

Theorem 1. *The map which sends $\widetilde{P}_\lambda(X)$ to τ_λ for all $\lambda \in \mathcal{D}_n$ and $\widetilde{P}_{n,n}(X)$ to q extends to a surjective* ring *homomorphism $\Lambda'_n \to QH^*(OG)$ with kernel generated by the relations $\widetilde{P}_{i,i}(X) = 0$ for $1 \leqslant i \leqslant n-1$. The ring $QH^*(OG)$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1,\ldots,\tau_n,q]$ modulo the relations*

$$\tau_i^2 + 2\sum_{k=1}^{i-1}(-1)^k\tau_{i+k}\tau_{i-k} + (-1)^i\tau_{2i} = 0 \tag{3}$$

for all $i < n$, together with the quantum relation

$$\tau_n^2 = q \tag{4}$$

(*it is understood that $\tau_j = 0$ for $j > n$*). *The Schubert class τ_λ in this presentation is given by the Giambelli formulas*

$$\tau_{i,j} = \tau_i\tau_j + 2\sum_{k=1}^{j-1}(-1)^k\tau_{i+k}\tau_{j-k} + (-1)^j\tau_{i+j} \tag{5}$$

for $i > j > 0$, and

$$\tau_\lambda = \text{Pfaffian}[\tau_{\lambda_i,\lambda_j}]_{1\leqslant i<j\leqslant r}, \tag{6}$$

where quantum multiplication is employed throughout. In other words, classical Giambelli and quantum Giambelli coincide for OG.

We remark that the statements in Theorem 1 are direct analogues of the corresponding facts for SL_N-Grassmannians [Be]. However, these results stand in contrast to the case of the Lagrangian Grassmannian $LG(n,2n)$, where quantum Giambelli does not coincide with classical Giambelli on $LG(n,2n)$ (see [KT2] for more details).

Our proof of Theorem 1 follows the scheme of [KT2], with two main differences. We require a Pfaffian identity for type D Schubert polynomials [KT1, §3.3], which gives a key relation in the Chow group of a certain *orthogonal Quot scheme* OQ_d. The latter scheme compactifies the moduli space of degree d maps $\mathbb{P}^1 \to OG$; however our definition of OQ_d differs from that in the Lagrangian case of [KT2], as the direct analogue of the Grothendieck Quot scheme [G1] here is not suitable for doing computations.

In $QH^*(OG)$ there are formulas

$$\tau_\lambda \cdot \tau_\mu = \sum \langle \tau_\lambda, \tau_\mu, \tau_{\widehat{\nu}} \rangle_d \, \tau_\nu \, q^d,$$

where the sum is over $d \geqslant 0$ and strict partitions ν with $|\nu| = |\lambda| + |\mu| - 2nd$, and $\widehat{\nu}$ is the dual partition of ν, whose parts complement the parts of ν in the set $\{1, \ldots, n\}$. Each quantum structure constant $\langle \tau_\lambda, \tau_\mu, \tau_{\widehat{\nu}} \rangle_d$ is a genus zero Gromov–Witten invariant for OG, and is a nonnegative integer. We present explicit formulas and algorithms to compute these numbers. This includes a quantum Pieri rule, which extends the classical result of Hiller and Boe [HB]. As an application, we show that there is a direct identification between the 3-point, genus zero Gromov–Witten invariants on OG with corresponding ones for the Lagrangian Grassmannian $LG(n-1, 2n-2)$ (Theorem 6).

This paper is organized as follows. In Section 2 we study the $\widetilde{P}$-polynomials and type D Schubert polynomials, and prove a remarkable Pfaffian identity for the latter. The orthogonal Grassmannians are introduced in Section 3, which includes a proof of the presentation for $QH^*(OG)$. The proof of the quantum Giambelli formula (6) of Theorem 1 is done in Sections 4 and 5, by studying intersections on the orthogonal Quot scheme. In Section 6 we formulate a 'quantum Schubert calculus' for OG. Finally, the Appendix establishes an identity for $\widetilde{P}$-polynomials which is used in [KT1].

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2. $\widetilde{P}$-polynomials and type D Schubert polynomials

2.1. **Basic definitions.** All the notational conventions used in this section follow [KT1] and [KT2]. In particular, for strict partitions λ and μ, the difference $\lambda \smallsetminus \mu$ denotes the partition with parts given by the parts of λ which are not parts of μ. A *composition* is a sequence of nonnegative integers with only finitely many nonzero parts. The $\widetilde{P}$-polynomials make sense when indexed by any composition ν, and satisfy Pfaffian relations

$$\widetilde{P}_\nu(X) = \sum_{j=1}^{g-1} (-1)^{j-1} \widetilde{P}_{\nu_j, \nu_g}(X) \cdot \widetilde{P}_{\nu \smallsetminus \{\nu_j, \nu_g\}}(X), \tag{7}$$

where g is an even number such that $\nu_i = 0$ for $i > g$. Define also the $\widetilde{Q}$-polynomial $\widetilde{Q}_\nu(X) = 2^\ell \, \widetilde{P}_\nu(X)$ for each composition ν with ℓ nonzero parts. The $\widetilde{Q}$-polynomials have integer coefficients, and span the ring $\mathbb{Z}[X]^{S_n}$ of symmetric functions in n variables.

Let $\widetilde{W}_n$ be the Weyl group for the root system D_n, whose elements are denoted as barred permutations. Recall that W_n is generated by the elements $s_\Box, s_1, \ldots, s_{n-1}$: for $i > 0$, s_i is the transposition interchanging i and $i+1$, and $s_\Box$ is defined by

$$(u_1, u_2, u_3, \ldots, u_n)s_\Box = (\overline{u}_2, \overline{u}_1, u_3, \ldots, u_n).$$

Let $\widetilde{w}_0$ denote the element of maximal length in $\widetilde{W}_n$. For each $\lambda \in \mathcal{D}_{n-1}$ we have a *maximal Grassmannian element* w_λ of $\widetilde{W}_n$, defined as in [KT1, §3.2].

Each generator s_i acts naturally on the polynomial ring $A[X]$, where $A = \mathbb{Z}[\frac{1}{2}]$; for $i > 0$, s_i interchanges x_i and x_{i+1}, while $s_\Box$ sends (x_1, x_2) to $(-x_2, -x_1)$; all other variables remain fixed. There are divided difference operators ∂'_i and $\partial_\Box$ on $A[X]$; for $i > 0$ they are defined by

$$\partial'_i(f) = (f - s_i f)/(x_{i+1} - x_i)$$

while

$$\partial_\Box(f) = (f - s_\Box f)/(x_1 + x_2),$$

for all $f \in A[X]$. These give rise to operators $\partial'_w : A[X] \to A[X]$ for each element $w \in \widetilde{W}_n$, as in [KT1, §3.2].

For all $w \in \widetilde{W}_n$ we have a *type D Schubert polynomial* $\mathfrak{D}_w(X) \in A[X]$ defined by

$$\mathfrak{D}_w(X) = (-1)^{n(n-1)/2} \partial'_{w^{-1}\widetilde{w}_0}\Big(x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \widetilde{P}_{n-1}(X)\Big).$$

These type D polynomials were defined in [KT1, §3.3]; they agree with the orthogonal Schubert polynomials of [LP] up to a sign, which depends on the degree. The polynomial $\mathfrak{D}_w(X)$ represents the Schubert class associated to w in the cohomology ring of the flag manifold SO_{2n}/B. Let us define $\mathfrak{D}'_\lambda(X) = \mathfrak{D}_{w_\lambda s_\Box}(X)$. It follows from the definitions and [KT1, Theorem 7] that $\mathfrak{D}'_\lambda(X) = \partial_\Box(\widetilde{P}_\lambda(X))$, for all non-zero partitions $\lambda \in \mathcal{D}_{n-1}$.

2.2. **A Pfaffian identity.** We require the identity in the following theorem for our proof of the quantum Giambelli formula for $OG(n+1, 2n+2)$.

Theorem 2. *Fix $\lambda \in \mathcal{D}_n$ of length $\ell \geqslant 3$, and set $r = 2\lfloor(\ell+1)/2\rfloor$. Then*

$$\sum_{j=1}^{r-1} (-1)^{j-1}\, \mathfrak{D}'_{\lambda_j, \lambda_r}(X)\, \mathfrak{D}'_{\lambda \smallsetminus \{\lambda_j, \lambda_r\}}(X) = 0. \tag{8}$$

Proof. We first observe, using the homogeneity of the two sides, that (8) is equivalent to the identity

$$\sum_{j=1}^{r-1} (-1)^{j-1}\, \partial_\Box(\widetilde{Q}_{\lambda_j, \lambda_r}(X)) \cdot \partial_\Box(\widetilde{Q}_{\lambda \smallsetminus \{\lambda_j, \lambda_r\}}(X)) = 0 \tag{9}$$

for $\widetilde{Q}$-polynomials, which should hold for λ and r as in the theorem.

Let $X'' = (x_3, \ldots, x_n)$ and define

$$m_{r,s}(x_1, x_2) = \begin{cases} x_1^r x_2^s + x_1^s x_2^r & \text{if } r \neq s, \\ x_1^r x_2^r & \text{if } r = s \end{cases}$$

to be the monomial symmetric function in x_1 and x_2. For any partition λ and nonnegative integers a and b, let $C(\lambda, a, b)$ denote the set of compositions μ with $\lambda_i - \mu_i \in \{0, 1, 2\}$ for all i and $\lambda_i - \mu_i = 1$ (resp. $\lambda_i - \mu_i = 2$) for exactly a (resp. b) values of i.

Proposition 1. *For any nonzero strict partition λ, we have*

$$\partial_\Box(\widetilde{Q}_\lambda(X)) = 2\sum_{\substack{0\leqslant s\leqslant r\leqslant \ell\\ r+s\ \text{even}}} m_{r,s}(x_1,x_2)\sum_{\substack{a+2b=r+s+1\\ 0\leqslant b\leqslant s}}\binom{a-1}{s-b}\sum_{\mu\in C(\lambda,a,b)}\widetilde{Q}_\mu(X''). \tag{10}$$

Proof. Let $X'=(x_2,\ldots,x_n)$. According to [KT2, Prop. 1], for any partition λ of length ℓ (not necessarily strict), we have

$$\widetilde{Q}_\lambda(X)=\sum_{k=0}^{\ell}x_1^k\sum_{\mu\in B(\lambda,k)}\widetilde{Q}_\mu(X'), \tag{11}$$

where $B(\lambda,k)$ is defined to be the set of all compositions μ such that $|\lambda|-|\mu|=k$ and $\lambda_i-\mu_i\in\{0,1\}$ for each i. By applying (11) twice we obtain

$$\widetilde{Q}_\lambda(X)=\sum_{0\leqslant s\leqslant r\leqslant\ell} m_{r,s}(x_1,x_2)\sum_{\substack{j+2k=r+s\\ 0\leqslant k\leqslant s}}\binom{j}{s-k}\sum_{\mu\in C(\lambda,j,k)}\widetilde{Q}_\mu(X''). \tag{12}$$

Suppose that $r\geqslant s\geqslant 0$. If $r+s$ is even, then $\partial_\Box(m_{r,s}(x_1,x_2))=0$. If $r+s$ is odd, we have

$$\partial_\Box(m_{r,s}(x_1,x_2))=2\sum_{\substack{c+d=r+s-1\\ c,d\geqslant s}}(-1)^{c-s}x_1^c x_2^d.$$

We now apply this to (12) and gather terms to obtain (10). $\Box$

Example. For all a, b with $a>b\geqslant 0$, we have

$$\begin{aligned}\partial_\Box(\widetilde{Q}_{a,b}(X)) &= 2\left(\widetilde{Q}_{a-1,b}(X'')+\widetilde{Q}_{a,b-1}(X'')\right)\\ &\quad+2\,x_1x_2\left(\widetilde{Q}_{a-2,b-1}(X'')+\widetilde{Q}_{a-1,b-2}(X'')\right).\end{aligned} \tag{13}$$

In the equation (13) and later on we agree that $\widetilde{Q}_\mu(X'')=0$ if any of the components of μ are negative.

As in [KT2, §2.3], the rest of the argument can be expressed using only the partitions which index the polynomials involved. We thus begin by defining a commutative $\mathbb{Z}$-algebra $\mathcal{B}$ with formal variables which represent these indices. The algebra $\mathcal{B}$ is generated by symbols $(a_1,a_2,\ldots)$, where the entries a_i are barred integers; each a_i can have up to two bars. The symbol $(a_1,a_2,\ldots)$ corresponds to the polynomial $\widetilde{Q}_\mu(X'')$, where μ is the composition with μ_i equal to the integer a_i minus the number of bars over a_i. We identify $(a,0)$ with (a).

Let μ be a barred partition, that is, a partition in which bars have been added to some of the entries. For $\ell(\mu)\geqslant 3$, we impose the Pfaffian relation

$$(\mu)=\sum_{j=1}^{m-1}(-1)^{j-1}(\mu_j,\mu_m)\cdot(\mu\smallsetminus\{\mu_j,\mu_m\}), \tag{14}$$

which corresponds to (7) for $\nu=\mu$ (here $m=2\lfloor(\ell(\mu)+1)/2\rfloor$, as usual). Iterating this gives

$$(\mu)=\sum\epsilon(\mu,\nu)(\nu_1,\nu_2)\cdots(\nu_{m-1},\nu_m), \tag{15}$$

where the sum is over all $(m-1)(m-3)\cdots(1)$ ways to write the set $\{\mu_1,\ldots,\mu_m\}$ as a union of pairs $\{\nu_1,\nu_2\}\cup\cdots\cup\{\nu_{m-1},\nu_m\}$, and where $\epsilon(\mu,\nu)$ is the sign of the permutation that takes $(\mu_1, \ldots, \mu_m)$ into $(\nu_1, \ldots, \nu_m)$; we adopt the convention that $\nu_{2i-1} \geqslant \nu_{2i}$.

We also define the square bracket symbols $[a] = (\overline{a})$ and $[a, b] = (\overline{a}, b) + (a, \overline{b})$, where a and b are integers, each with up to one bar. For example, the right hand side of equation (13) corresponds to the sum $2\,[a, b] + 2\,x_1x_2\,[\overline{a}, \overline{b}]$ in $\mathcal{B}[x_1, x_2]$. Finally, we impose the relations

$$[a, b] = (\overline{a})(b) - (a)(\overline{b}) \tag{16}$$

for integers a, b, with up to one bar each; this agrees with a corresponding identity

$$\widetilde{Q}_{a-1,b} + \widetilde{Q}_{a,b-1} = \widetilde{Q}_{a-1}\widetilde{Q}_b - \widetilde{Q}_a\widetilde{Q}_{b-1}$$

of $\widetilde{Q}$-polynomials.

Using these conventions and equations (10) and (13), we are reduced to showing that $S_1 + S_2 = 0$, where

$$S_1 = \sum_{\substack{a+2b=r+s+1 \\ 0\leqslant b\leqslant s}} \binom{a-1}{s-b} \sum_{j=1}^{r-1} (-1)^{j-1}\,[\lambda_j, \lambda_r] \sum_{\mu\in C(\lambda\smallsetminus\{\lambda_j,\lambda_r\},a,b)} (\mu),$$

$$S_2 = \sum_{\substack{a'+2b'=r+s-1 \\ 0\leqslant b'\leqslant s-1}} \binom{a'-1}{s-b'-1} \sum_{j=1}^{r-1} (-1)^{j-1}\,[\overline{\lambda}_j, \overline{\lambda}_r] \sum_{\mu\in C(\lambda\smallsetminus\{\lambda_j,\lambda_r\},a',b')} (\mu),$$

and $r \geqslant s \geqslant 0$ are fixed integers with $r + s$ even. The proof of this is rather similar to the proofs of Theorems 2 and 3 of [KT2], and we will point out only the main difference here.

We first apply (15) to expand the terms (μ) in both S_1 and S_2. The cancellation technique of [KT2, §2.3], notably, the identity

$$[a, b][c, d] - [a, c][b, d] + [a, d][b, c] = 0, \tag{17}$$

implies the vanishing of the sum of those summands in S_1 which contain a pair with exactly one bar, or at least two pairs with exactly three bars. The remainder is a sum S_1' consisting of those summands in S_1 with a unique pair containing three bars, and no pair with only one bar. In the same way, one checks the vanishing of the sum of those summands in S_2 which contain a pair with exactly three bars, or at least two pairs with exactly one bar. There remains a sum S_2' consisting of those summands in S_2 with a unique pair containing only one bar, and no pair with exactly three bars. Hence, it is enough to show that $S_1' + S_2' = 0$.

There is an obvious bijection between the summands in S_1' and S_2', obtained by adding two bars to the unbarred part of the pair in S_2' which contains only one bar (note that the corresponding binomial coefficients agree, as $(a, b) = (a', b' + 1)$ for these two summands). To prove that the sum of all corresponding terms is zero, it suffices to show that the expression

$$\Big([a, b][\overline{c}, \overline{d}] - [a, c][\overline{b}, \overline{d}] + [a, d][\overline{b}, \overline{c}]\Big) + \Big([\overline{a}, \overline{b}][c, d] - [\overline{a}, \overline{c}][b, d] + [\overline{a}, \overline{d}][b, c]\Big) \tag{18}$$

vanishes identically in $\mathcal{B}$ (we then apply this with $a = \lambda_r$, always). To check this, begin from the basic identities

$$[a, b][\overline{c}, \overline{d}] - [a, \overline{c}][b, \overline{d}] + [a, \overline{d}][b, \overline{c}] = 0 \tag{19}$$

and

$$[\overline{a},\overline{b}][c,d]-[\overline{a},c][\overline{b},d]+[\overline{a},d][\overline{b},c]=0 \tag{20}$$

which are easily shown using (16). Let $\langle x,y\rangle=[\overline{x},y]+[x,\overline{y}]$ and note that

$$\langle a,b\rangle\,\langle c,d\rangle-\langle a,c\rangle\,\langle b,d\rangle+\langle a,d\rangle\,\langle b,c\rangle=0, \tag{21}$$

which is shown using $\langle x,y\rangle=(\overline{\overline{a}})(b)-(a)(\overline{\overline{b}})$ (another consequence of (16)). The vanishing of (18) follows by combining (19), (20) and (21). □

3. Orthogonal Grassmannians

3.1. **Schubert varieties and incidence loci.** Let V be a fixed $(2n+2)$-dimensional complex vector space equipped with a nondegenerate symmetric bilinear form on V. The principal object of study is the orthogonal Grassmannian $OG(n+1,2n+2)$ which is one component of the parameter space of $(n+1)$-dimensional isotropic subspaces of V. When n is fixed, we write OG for $OG(n+1,2n+2)$. We have $\dim_{\mathbb{C}} OG=n(n+1)/2$. The identities in cohomology that we establish in this section remain valid if we work over an arbitrary base field, and use Chow rings in place of cohomology.

Let $F_\bullet$ be a fixed complete isotropic flag of subspaces of V. By convention, then, OG parametrizes maximal isotropic spaces $\Sigma\subset V$ such that $\Sigma\cap F_{n+1}$ has even codimension in F_{n+1}. We define the alternative flag $\widetilde{F}_\bullet$ to be the flag $F_1\subset\cdots\subset F_n\subset\widetilde{F}_{n+1}$, where $\widetilde{F}_{n+1}$ is the unique maximal isotropic space containing F_n but not equal to F_{n+1}. We let

$$F_\bullet^{(i)}=\begin{cases}F_\bullet & \text{if } i\equiv(n+1)\bmod 2,\\ \widetilde{F}_\bullet & \text{otherwise.}\end{cases} \tag{22}$$

The Schubert varieties $\mathfrak{X}_\lambda\subset OG$ are indexed by partitions $\lambda\in\mathcal{D}_n$. We record two ways to write the conditions which define the Schubert variety $\mathfrak{X}_\lambda$:

$$\mathfrak{X}_\lambda=\{\,\Sigma\in OG\mid \operatorname{rk}(\Sigma\to V/F_{n+1-\lambda_i})\leqslant n+1-i,\ i=1,\ldots,\ell(\lambda)\,\} \tag{23}$$

$$=\{\,\Sigma\in OG\mid \operatorname{rk}(\Sigma\to V/F^{(i)\perp}_{n+1-\lambda_i})\leqslant n+1-i-\lambda_i,\ i=1,\ldots,\ell(\lambda)+1\,\}. \tag{24}$$

Let τ_λ be the class of $\mathfrak{X}_\lambda$ in $H^*(OG,\mathbb{Z})$. The classical Giambelli formula (6) for OG is equivalent to the following identity in $H^*(OG,\mathbb{Z})$:

$$\tau_\lambda=\sum_{j=1}^{r-1}(-1)^{j-1}\tau_{\lambda_j,\lambda_r}\cdot\tau_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}}, \tag{25}$$

for $r=2\lfloor(\ell(\lambda)+1)/2\rfloor$. Let $\rho_n=(n,n-1,\ldots,1)$ and for $\mu\in\mathcal{D}_n$, denote by $\widehat{\mu}=\rho_n\smallsetminus\mu$, the dual partition. The Poincaré duality pairing on OG satisfies

$$\int_{OG}\tau_\lambda\,\tau_\mu=\delta_{\lambda\widehat{\mu}}.$$

Given an isotropic space $A\subset V$ of dimension $n-k$ ($k\geqslant 0$), the variety of maximal isotropic spaces containing A is a translate of the Schubert variety $\mathfrak{X}_{n,n-1,\ldots,k+1}$. We have the following result on intersections of such varieties with the Schubert varieties $\mathfrak{X}_\lambda$; this is analogous to a similar result in type C ([KT2, Prop. 3]).

Proposition 2. *Let $k \geqslant 0$ and $\lambda \in \mathcal{D}_n$. Let A be an isotropic subspace of V of dimension $n-k$, and let $Y \subset OG$ be the subvariety of maximal isotropic subspaces of V which contain A. Then $\mathfrak{X}_\lambda \cap Y$ is a Schubert variety in $Y \simeq OG(k+1, 2k+2)$. Moreover, if $\ell(\lambda) < k$ then the intersection, if nonempty, has positive dimension.*

Proof. As in [KT2], the intersection is defined by the attitude of Σ/A with respect to $F'_\bullet$, where $F'_i = ((F_i + A) \cap A^\perp)/A$. For the intersection to be a point would require at least k rank conditions, and hence $\ell(\lambda) \geqslant k$. □

The space $OG(n-1, 2n+2)$ is the parameter space of lines on OG. For a nonempty partition λ, the variety of lines incident to $\mathfrak{X}_\lambda$ is the Schubert variety $\mathfrak{Y}_\lambda$, consisting of those $\Sigma' \in OG(n-1, 2n+2)$ such that

$$\text{rk}(\Sigma' \to V/F^{(i)\perp}_{n+1-\lambda_i}) \leqslant n+1-i-\lambda_i, \quad \text{for} \quad i = 1, \ldots, \ell+1. \tag{26}$$

The codimension of $\mathfrak{Y}_\lambda$ is $|\lambda|-1$. Note that (i) the rank conditions (26) are identical to those in (24); (ii) the rank condition corresponding to $i = \ell(\lambda)+1$, which was redundant in defining the Schubert varieties in OG, is necessary here.

3.2. **A Pfaffian identity on $OG(n-1, 2n+2)$.** Let $F = F_{SO}(V)$ denote the variety of complete isotropic flags in $V = \mathbb{C}^{2n+2}$. There is a natural projection map from F to the orthogonal Grassmannian $OG(n-1, 2n+2)$, inducing an injective pullback morphism on cohomology. Introduce an extra variable x_{n+1} and let $X^+ = (x_1, \ldots, x_{n+1})$. Referring to [KT1, §2.4 and Sect. 3], we check that the Schubert class $[\mathfrak{Y}_\lambda]$ in $H^*(OG(n-1, 2n+2))$ pulls back to the class represented by $\mathfrak{D}'_\lambda(X^+)$ in $H^*(F)$, for each $\lambda \in \mathcal{D}_{n-1}$. Here X^+ corresponds to the vector of Chern roots of the dual to the tautological rank $n+1$ vector bundle over F, ordered as in [KT1, Sect. 2]. Theorem 2 remains true with X^+ in place of X, and gives

Corollary 1. *For every $\lambda \in \mathcal{D}_n$ of length $\ell \geqslant 3$ and $r = 2\lfloor(\ell+1)/2\rfloor$ we have*

$$\sum_{j=1}^{r-1} (-1)^{j-1} \, [\mathfrak{Y}_{\lambda_j, \lambda_r}] \, [\mathfrak{Y}_{\lambda \smallsetminus \{\lambda_j, \lambda_r\}}] = 0 \tag{27}$$

in $H^*(OG(n-1, 2n+2), \mathbb{Z})$.

3.3. **Quantum relations and two-condition Giambelli.** Recall that in $QH(OG)$, the degree of q is

$$\int_{OG} c_1(T_{OG}) \cdot \tau_{\hat{1}} = 2n.$$

It follows, for degree reasons, that the relations in cohomology (3) and the quantum Giambelli formula for the two-condition Schubert classes (5) – which we know to hold classically – hold in $QH(OG)$. The degree $2n$ quantum relation (4) follows from the elementary enumerative fact that there is a unique line on OG through a given point, incident to two general translates of $\mathfrak{X}_n$. Arguing as in [ST], now, we obtain a presentation of $QH^*(OG)$ as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \ldots, \tau_n, q]$ modulo the relations (3) and (4) (see also [FP, Sect. 10]).

The proof of the more difficult quantum Giambelli formula (6) occupies Sections 4 and 5.

4. Orthogonal Quot schemes

4.1. **Overview.** In the next two sections, we define the orthogonal Quot scheme and establish an identity in its Chow group, from which identity (6) in $QH^*(OG)$ readily follows. We make use of type D degeneracy loci for isotropic morphisms of vector bundles [KT1] to define classes $[W_\lambda(p)]_k$ ($p \in \mathbb{P}^1$) of the appropriate dimension $k := n(n+1)/2 + 2nd - |\lambda|$ in the Chow group of the orthogonal Quot scheme OQ_d, which compactifies the space of degree-d maps $\mathbb{P}^1 \to OG$. Let $p' \in \mathbb{P}^1$ be distinct from p, and denote by W' the degeneracy locus defined by a general translate of the fixed isotropic flag $F_\bullet$. We produce a Pfaffian formula analogous to (25):

$$[W_\lambda(p)]_k = \sum_{j=1}^{r-1} (-1)^{j-1} [W_{\lambda_j,\lambda_r}(p) \cap W'_{\lambda \smallsetminus \{\lambda_j,\lambda_r\}}(p')]_k, \tag{28}$$

for any $\lambda \in \mathcal{D}_n$ with $\ell(\lambda) \geqslant 3$ and $r = 2\lfloor (\ell(\lambda)+1)/2 \rfloor$.

As in [KT2], we need the cycles in (28) to remain rationally equivalent under further intersection with some (general translate of) $W_\mu(p'')$, for $\mu \in \mathcal{D}_n$ and $p'' \in \mathbb{P}^1$ distinct from p, p', Also, as in loc. cit., we accomplish this by working on a modification $OQ_d(p'')$, on which the evaluation-at-p'' map is globally defined, and employing refined intersection operation from OG.

The rational equivalences that we produce — (28) and a similar equivalence on $OQ_d(p'')$ — come by combining equivalences of the following types: (i) the classical Pfaffian formulas on OG (25); (ii) the Pfaffian identities (27) on $OG(n-1, 2n+2)$; (iii) rational equivalences $\{p\} \sim \{p'\}$ on $\mathbb{P}^1$. Indeed, the essence of (iii) is that we can replace p' with p in (28); the intersection $W_{\lambda_j,\lambda_r}(p) \cap W'_{\lambda \smallsetminus \{\lambda_j,\lambda_r\}}(p)$ now has k-dimension components supported in the boundary of the Quot scheme. The cancellation of these contributions in the Chow group is precisely equation (27).

4.2. **Definition of OQ_d.** Let V be a complex vector space V of dimension $N = r + s$ and fix $d \geqslant 0$. Following Grothendieck [G1], there is a smooth projective variety Q_d, the *Quot scheme*, which parametrizes flat families of quotient sheaves of $\mathcal{O}_{\mathbb{P}^1} \otimes V$ with Hilbert polynomial $p(t) = st + s + d$. This variety compactifies the space of parametrized degree-d maps from $\mathbb{P}^1$ to the Grassmannian of r-dimensional subspaces of V. On $\mathbb{P}^1 \times Q_d$ there is a universal exact sequence of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O} \otimes V \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{29}$$

with $\mathcal{E}$ locally free of rank r. From now on, we fix V as in Section 3 and $r = s = n+1$.

Definition 1. Let d be a nonnegative integer. The *isotropic locus* Q_d^{iso} is the closed subscheme of Q_d which is defined by the vanishing of the composite

$$\mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \otimes V \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^1} \otimes V^* \longrightarrow \mathcal{E}^*$$

where α is the isomorphism defined by the given bilinear form on V.

The embedding of OG in the Grassmannian $G(n+1, 2n+2)$ of $(n+1)$-dimensional subspaces of V is degree-doubling, that is, in the sheaf sequence (29) corresponding to degree-d maps $\mathbb{P}^1 \to OG$, the sheaf $\mathcal{Q}$ has degree $2d$. For any d, Q_{2d}^{iso} contains an open subscheme isomorphic to the moduli space $M_{0,3}(OG, d)$:

Definition 2. Let d be a nonnegative integer. Then OM_d is the open subscheme of Q_{2d}^{iso} defined by the conditions (i) $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1} \otimes V$ has everywhere full rank; (ii) the image of $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1} \otimes V$ at any point has intersection with F_{n+1} of dimension congruent to $(n+1)$ mod 2.

Unfortunately, Q_{2d}^{iso} generally has components of dimension larger than the dimension of OM_d. The remedy is to throw away any point of (29) where the rank of $\mathcal{E} \to \mathcal{O} \otimes V$ drops by just 1 at some point of $\mathbb{P}^1$. We can do this, and still be left with a closed subscheme of Q_{2d}^{iso}, because in any degeneration situation in which the rank of $\mathcal{E} \to \mathcal{O} \otimes V$ drops from full to less than full, the drop is by at least 2.

Definition 3. For $d \in (1/2)\mathbb{Z}$, the *orthogonal Quot scheme* OQ_d is the subset of Q_{2d}^{iso} consisting of points whose sheaf sequence (29) satisfies $\mathrm{rk}(\mathcal{E}_p \to V) \neq n$ for all $p \in \mathbb{P}^1$, and such that where it has full rank, the image has intersection with F_{n+1} of even codimension in F_{n+1}. This subset, evidently constructible and closed by virtue of Proposition 3, below, is given the reduced scheme structure.

Lemma 1. *Let* $\psi\colon C_0 \to G(n+1, 2n+2)$ *be a morphism, with* $C_0 \cong \mathbb{P}^1$*, and let* C *be a tree of* $\mathbb{P}^1$*'s containing* C_0 *and* $\varphi\colon C \to G(n+1, 2n+2)$ *a map which restricts to* ψ *on* C_0*. Let*

$$\widetilde{C} := C_1 \cup C_2 \cup \cdots \cup C_m$$

$(m \geqslant 1)$ *denote a chain of components in* C*, with* $C_i \neq C_0$ *for all* $i \geqslant 1$*, and assume* C_1 *meets* C_0 *at the point* p *and* C_i *is collapsed by* φ *for all* i *with* $1 \leqslant i \leqslant m-1$*. Let* $\pi\colon C \to C_0$ *denote the morphism which collapses all components of* C *except* C_0*. Let*

$$0 \to \mathcal{E}_0 \to \mathcal{O} \otimes V \to \mathcal{Q}_0 \to 0$$

denote the pullback of the universal sequence via ψ*, and let*

$$0 \to \mathcal{E} \to \mathcal{O} \otimes V \to \mathcal{Q} \to 0$$

denote the pullback of the universal sequence via φ *(so that* $\mathcal{E}|_{C_0} \simeq \mathcal{E}_0$*). Assume the restriction of* $\mathcal{E}$ *to* C_m *splits as*

$$\mathcal{O}(-b_1) \oplus \cdots \oplus \mathcal{O}(-b_j) \oplus \mathcal{O}^{n+1-j}$$

with $b_1, \ldots, b_j \geqslant 1$*. Then the morphism* $\pi_*\mathcal{E} \to \pi_*(\mathcal{O} \otimes V) = \mathcal{O} \otimes V$ *factors through* $\mathcal{E}_0$*, and the cokernel of* $\pi_*\mathcal{E} \to \mathcal{E}_0$ *is a torsion sheaf whose fiber at* p *has dimension at least* j*.*

Proof. We may choose $n-j$ independent sections $s_1, \ldots, s_{n-j}$ of $\mathcal{E}|_{C_m}$. These extend uniquely to $n-j$ independent sections of $\mathcal{E}|_{\widetilde{C}}$, and hence span an $(n-j)$-dimensional subspace Σ of the fiber of $\mathcal{E}$ at the point p. The map $(\pi_*\mathcal{E})_p \to (\mathcal{E}_0)_p$ on fibers at p has image contained in Σ. Hence the dimension of the fiber at p of the cokernel of $\pi_*\mathcal{E} \to \mathcal{E}_0$ is at least j. □

Proposition 3. *For any* $d \in (1/2)\mathbb{Z}$*, the subset* $OQ_d \subset Q_{2d}^{\mathrm{iso}}$ *is closed under specialization.*

Proof. Suppose $x_1 \in OQ_d$ specializes to $x_0 \in Q_{2d}$. Then there is a discrete valuation ring R and a morphism $\varphi\colon \operatorname{Spec} R \to Q_{2d}$ such that the generic point maps to x_1 and the special point maps to x_0.

Denote the fraction field of R by K and the residue field by k. It suffices to consider the case where x_0 is a closed point, hence $k = \mathbb{C}$ is algebraically closed. We show that given the exact sequence of coherent sheaves at the generic point

$$0 \to \mathcal{E} \to \mathcal{O} \otimes V \to \mathcal{Q} \to 0 \tag{30}$$

on $\mathbb{P}^1_K$, we can reconstruct the map φ and hence the sheaf sequence at the special point (possibly replacing R by its integral closure in a finite extension of K). Then, we note that the torsion of the quotient sheaf at the special point cannot have rank 1 at any point of $\mathbb{P}^1_k$.

Let the sequence (30) be given. The support of $\mathcal{Q}^{\mathrm{tors}}$ specializes to a well-defined closed subset $Z \subset \mathbb{P}^1_k$; we let $Y = \mathrm{Supp}(\mathcal{Q}^{\mathrm{tors}}) \cup Z$. Now consider:

$$0 \to \mathcal{E}' \to \mathcal{O} \otimes V \to \mathcal{Q}/\mathcal{Q}^{\mathrm{tors}} \to 0 \tag{31}$$

on $\mathbb{P}^1_K$. This corresponds to a morphism $\mathbb{P}^1_K \to OG$ (the actual map to the orthogonal Grassmannian underlying the sheaf sequence (30)). By replacing K by a finite extension and R by its integral closure in the extension, if necessary, then there exists, by semistable reduction, a modification

$$\pi\colon S \to \mathbb{P}^1_R$$

with exceptional divisor a tree of $\mathbb{P}^1$'s, and a morphism $S \to OG$, such that π restricts to the given morphism $\mathbb{P}^1_K \to OG$. We consider the pullback of the universal exact sequence

$$0 \to \widetilde{\mathcal{E}} \to \mathcal{O} \otimes V \to \widetilde{\mathcal{Q}} \to 0$$

on S. Pushing forward the map $\mathcal{E} \to \mathcal{O} \otimes V$ by π yields an exact sequence

$$0 \to \pi_*\widetilde{\mathcal{E}} \to \mathcal{O} \otimes V \to \mathcal{C} \to 0 \tag{32}$$

The cokernel $\mathcal{C}$, being a subsheaf of $\pi_*\widetilde{\mathcal{Q}}$, is torsion-free over $\operatorname{Spec} R$, and hence flat: (32) corresponds to the map from $\operatorname{Spec} R$ to the (possibly smaller degree) Quot scheme determined by (31).

We extend (30) to all of $\mathbb{P}^1_R$ by patching and pushing forward. The sequences (30) on $\mathbb{P}^1_K$ and (32) on $\mathbb{P}^1_R \smallsetminus Y$ patch to give the sequence

$$0 \to \widehat{\mathcal{E}} \to \mathcal{O} \otimes V \to \widehat{\mathcal{Q}} \to 0$$

on $\mathbb{P}^1_R \smallsetminus Z$. Pushing forward via $i\colon \mathbb{P}^1_R \smallsetminus Z \to \mathbb{P}^1_R$ gives

$$0 \to i_*\widehat{\mathcal{E}} \to \mathcal{O} \otimes V \to \mathcal{D} \to 0, \tag{33}$$

(where $\mathcal{D}$ is the indicated cokernel), flat over $\mathbb{P}^1_R$ since $i_*\widehat{\mathcal{E}}$ is locally free. This gives the morphism $\varphi\colon \operatorname{Spec} R \to Q_{2d}$ that we started with.

We now consider the restriction of (33) to the special fiber:

$$0 \to (i_*\widehat{\mathcal{E}})_k \to \mathcal{O} \otimes V \to \mathcal{D}_k \to 0,$$

and verify it satisfies the rank conditions. By semicontinuity, the dimension of the fiber of $\mathcal{D}_k^{\mathrm{tors}}$ is $\geqslant 2$ at every point of Z. Suppose p is a point in $\mathbb{P}^1_k \smallsetminus Z$. Then $\mathcal{D}_k$, on a neighborhood of p, is isomorphic to $\mathcal{C}_k := \mathcal{C} \otimes_R k$, so it suffices to show every nonzero fiber of $\mathcal{C}_k^{\mathrm{tors}}$ has dimension $\geqslant 2$. Letting $(\,)_k$ denote restriction to the special fiber, we have: $(\pi_*\widetilde{\mathcal{E}})_k \to \mathcal{O} \otimes V$ factors through $(\pi_k)_*(\widetilde{\mathcal{E}}_k) \to \mathcal{O} \otimes V$, which in turn factors through a vector subbundle $[(\pi_k)_*(\widetilde{\mathcal{E}}_k)]'$ of $\mathcal{O} \otimes V$ (the pullback of the universal subbundle by the actual map $\mathbb{P}^1_k \to OG$ at the special fiber), and $\dim \mathcal{C}_k^{\mathrm{tors}} \otimes \mathcal{O}_p$ is greater than or equal to the dimension of the fiber at p of $[(\pi_k)_*(\widetilde{\mathcal{E}}_k)]'/(\pi_k)_*(\widetilde{\mathcal{E}}_k)$. But now we are in the situation of Lemma 1: this dimension is at least the number of negative line bundles in the direct sum decomposition of the pullback of the universal subbundle of OG under some positive-degree map from a copy of $\mathbb{P}^1_k$ to OG, and this must be at least 2. □

4.3. **Degeneracy loci.** Degeneracy loci for vector bundles in type D were defined using rank inequalities in [KT1].

Definition 4. The degeneracy loci W_λ and $W_\lambda(p)$ ($\lambda \in \mathcal{D}_n$, with $\ell = \ell(\lambda)$, and $p \in \mathbb{P}^1$) are the following subschemes of $\mathbb{P}^1 \times OQ_d$:

$$W_\lambda = \{\, x \in \mathbb{P}^1 \times OQ_d \mid \mathrm{rk}(\mathcal{E} \to \mathcal{O} \otimes V/F^{(i)\perp}_{n+1-\lambda_i})_x \leqslant n+1-i-\lambda_i, i = 1, \dots, \ell+1 \,\},$$
$$W_\lambda(p) = W_\lambda \cap (\{p\} \times OQ_d)$$

Define also

$$h(n,d) = n(n+1)/2 + 2nd,$$

which is the dimension of the orthogonal Quot scheme OQ_d when d is a nonnegative integer. As in types A and C, we establish a Moving Lemma, and deduce from this that all three-term Gromov–Witten invariants on OG count points in intersections of degeneracy loci on OQ_d.

Moving Lemma. *Let k be a positive integer, and let $p_1, \dots, p_k$ be distinct points on $\mathbb{P}^1$. Let $\lambda^1, \dots, \lambda^k$ be partitions in $\mathcal{D}_n$, and let us take the degeneracy loci $W_{\lambda^1}(p_1), \dots, W_{\lambda^k}(p_k)$ to be defined by isotropic flags of vector spaces in general position. Consider the intersection*

$$Z := W_{\lambda^1}(p_1) \cap \cdots \cap W_{\lambda^k}(p_k).$$

Then Z has dimension at most $h(n,d) - \sum_{i=1}^k |\lambda^i|$. Moreover, $Z \cap OM_d$ is either empty or generically reduced and of pure dimension $h(n,d) - \sum_i |\lambda^i|$; also, $Z \cap (OQ_d \smallsetminus OM_d)$ has dimension at most $h(n,d) - \sum_{i=1}^k |\lambda^i| - 1$.

The following are immediate consequences of the Moving Lemma.

Corollary 2. *Let p, p', $p'' \in \mathbb{P}^1$ be distinct points. Suppose λ, μ, $\nu \in \mathcal{D}_n$ satisfy $|\lambda| + |\mu| + |\nu| = h(n,d)$. With degeneracy loci defined with respect to isotropic flags in general position, the intersection $W_\lambda(p) \cap W_\mu(p') \cap W_\nu(p'')$ consists of finitely many reduced points, all contained in OM_d, and the corresponding Gromov–Witten invariant on OG satisfies*

$$\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d = \#\big(W_\lambda(p) \cap W_\mu(p') \cap W_\nu(p'')\big).$$

Corollary 3. *If p and p' are distinct points of $\mathbb{P}^1$ and if $|\lambda| + |\mu| = h(n,d)$, then $W_\lambda(p) \cap W'_\mu(p') = \emptyset$ for a general translate $W'_\mu(p')$ of $W_\mu(p')$.*

The Moving Lemma itself is proved using an analysis of the boundary of OQ_d. As in [Be] and [KT2], this boundary is covered by Grassmann bundles over smaller Quot schemes.

Definition 5. For $c \in (1/2)\mathbb{Z}$, with $c \geqslant 1$, we let $\pi_c \colon G_c \to \mathbb{P}^1 \times OQ_{d-c}$ denote the Grassmann bundle of $(2c)$-dimensional quotients of the universal bundle $\mathcal{E}$ on $\mathbb{P}^1 \times OQ_{d-c}$. The morphism $\beta_c \colon G_c \to OQ_d$ is given by the modification of the sheaf sequence $\mathcal{E} \to \mathcal{O} \otimes V$ along the graph of the projection to $\mathbb{P}^1$. Precisely: let $\mathcal{F}_c$ denote the universal quotient bundle on G_c; if i_c denotes the morphism $G_c \to \mathbb{P}^1 \times G_c$ given by $(\mathrm{pr}_1 \circ \pi_c, \mathrm{id})$, then $\mathcal{E}_c$ is defined as the kernel of the natural morphism of sheaves $(\mathrm{id} \times (\mathrm{pr}_2 \circ \pi_c))^*\mathcal{E} \to i_{c*}\pi_c^*\mathcal{E}$ composed with i_{c*} applied to the morphism to $\mathcal{F}_c$.

We also consider degeneracy loci with respect to the bundles $\mathcal{E}_c$.

Definition 6. We define $\widehat{W}_{c,\lambda}$ and $\widehat{W}_{c,\lambda}(p)$ to be the following subschemes of G_c:

$$\widehat{W}_{c,\lambda} = \{\, x \in G_c \mid \operatorname{rk}(\mathcal{E}_c \to \mathcal{O} \otimes V/F^{\perp}_{n+1-\lambda_i})_x \leqslant n+1-i-\lambda_i,\ i = 1,\ldots,\ell+1 \,\},$$
$$\widehat{W}_{c,\lambda}(p) = \widehat{W}_{c,\lambda}(p) \cap \pi_c^{-1}(\{p\} \times OQ_{d-c})$$

4.4. **Boundary structure of OQ_d.** The boundary of OQ_d is made up of points where $\mathcal{E} \to \mathcal{O} \otimes V$ drops rank at one or more points of $\mathbb{P}^1$; note that wherever it drops rank, it does so by at least two (by our definition of the Quot scheme).

Theorem 3. *For any $d \in (1/2)\mathbb{Z}$, with $d \geqslant 0$ and $d \neq 1/2$, we have*

$$\dim OQ_d = \begin{cases} h(n,d) & \text{if } d \in \mathbb{Z}, \\ h(n,d) - 5 & \text{otherwise.} \end{cases}$$

Furthermore, for $c \in (1/2)\mathbb{Z}$, $c \geqslant 1$, the map $\beta_c \colon G_c \to OQ_d$ satisfies
(i) *Given $x \in OQ_d$, if $\mathcal{Q}_x$ has rank at least $n+1+c$ at $p \in \mathbb{P}^1$, then x lies in the image of β_c.*
(ii) *The restriction of β_c to $\pi_c^{-1}(\mathbb{P}^1 \times OM_{d-c})$ is a locally closed immersion.*
(iii) *We have*

$$\beta_c^{-1}(W_\lambda(p)) = \pi_c^{-1}(\mathbb{P}^1 \times W_\lambda(p)) \cup \widehat{W}_{c,\lambda}(p)$$

where on the right, $W_\lambda(p)$ denotes the degeneracy locus in OQ_{d-c}.

The proof of Theorem 3, as well as that of the Moving Lemma (which uses Theorem 3), is similar to that of the corresponding results in [Be] and [KT2]. Details are left to the reader.

5. Intersection Theory on OQ_d

The Chow group of algebraic cycles modulo rational equivalence of a scheme $\mathfrak{X}$ is denoted $A_*\mathfrak{X}$. We also employ the following notation.

Definition 7. Let p denote a point of $\mathbb{P}^1$.
(i) $\mathrm{ev}^p \colon OM_d \to OG$ is the evaluation at p morphism;
(ii) $\tau(p) \colon OQ_d(p) \to OQ_d$ is the projection from the *relative orthogonal Grassmannian* $OQ_d(p) := OG_{n+1}(\mathcal{Q}|_{\{p\}\times OQ_d})$, that is, the closed subscheme of the Grassmannian Grass_{n+1} of rank-$(n+1)$ quotients [G2] of the indicated coherent sheaf, defined by isotropicity and parity conditions on the kernel of the composite morphism from $\mathcal{O}_{\mathrm{Grass}_{n+1}} \otimes V$ to the universal quotient bundle of the relative Grassmannian;
(iii) $\mathrm{ev}(p) \colon OQ_d(p) \to LG$ is the evaluation morphism on the relative orthogonal Grassmannian;
(iv) $\mathrm{ev}^p_c \colon \pi_c^{-1}(\{p\} \times OM_{d-c}) \to OG(n+1-2c, 2n+2)$ is evaluation at p.

Lemma 2 ([KT2])**.** *Let T be a projective variety which is a homogenous space for an algebraic group G. Let $\mathfrak{X}$ be a scheme, equipped with an action of the group G. Let U be a G-invariant integral open subscheme of $\mathfrak{X}$, and let $f \colon U \to T$ be a G-equivariant morphism. Then the map on algebraic cycles*

$$[V] \mapsto \left[\, \overline{f^{-1}(V)} \,\right]$$

respects rational equivalence, and hence induces a map on Chow groups $A_*T \to A_*\mathfrak{X}$.

Corollary 4. *Fix distinct points p, $p' \in \mathbb{P}^1$. For any $\lambda \in \mathcal{D}_n$ of length $\ell = \ell(\lambda) \geqslant 3$, the following cycles are rationally equivalent to zero on OQ_d and on $OQ_d(p')$:*
(i) $\big[\,(\mathrm{ev}^p)^{-1}(\mathfrak{X}_\lambda)^-\,\big] - \sum_{j=1}^{r-1}(-1)^{j-1}\big[\,(\mathrm{ev}^p)^{-1}(\mathfrak{X}_{\lambda_j,\lambda_r} \cap \mathfrak{X}'_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}})^-\,\big]$.
(ii) $\sum_{j=1}^{r-1}(-1)^{j-1}\big[\,\beta_1\big((\mathrm{ev}_1^p)^{-1}(\mathfrak{Y}_{\lambda_j,\lambda_r} \cap \mathfrak{Y}'_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}})\big)^-\,\big]$.

Here, and in the sequel, $\mathfrak{X}'_\mu$ and $\mathfrak{Y}'_\mu$ denote the translates of $\mathfrak{X}_\mu$ and $\mathfrak{Y}_\mu$ by a general element of the group SO_{2n+2}.

As is standard, for any closed subscheme Z of a scheme $\mathfrak{X}$, $[Z] \in A_*\mathfrak{X}$ denotes the class in the Chow group of the cycle associated to Z; we let $[Z]_k$ be the dimension k component of $[Z]$.

Proposition 4. (a) *Suppose λ and μ are in $\mathcal{D}_n$, and let p, p', p'' be distinct points in $\mathbb{P}^1$. Assume that $\ell(\lambda)$ equals 1 or 2 and μ has even length $\geqslant 2$. Let $k = h(n,d) - |\lambda| - |\mu|$. Then*

$$\big[W_\lambda(p) \cap W'_\mu(p')\big]_k = \big[W_\lambda(p) \cap W'_\mu(p)\big]_k \text{ in } A_*OQ_d,$$
$$\big[\tau(p'')^{-1}\big(W_\lambda(p) \cap W'_\mu(p')\big)\big]_k = \big[\tau(p'')^{-1}\big(W_\lambda(p) \cap W'_\mu(p)\big)\big]_k \text{ in } A_*OQ_d(p''),$$

where $W'_\mu(p)$ denotes degeneracy locus with respect to a general translate of the isotropic flag of subspaces.

(b) *In A_*OQ_d, we have*

$$\big[W_\lambda(p) \cap W'_\mu(p)\big]_k = \big[\,(\mathrm{ev}^p)^{-1}(\mathfrak{X}_\lambda \cap \mathfrak{X}'_\mu)^-\,\big] + \big[\,\beta_1\big((\mathrm{ev}_1^p)^{-1}(\mathfrak{Y}_\lambda \cap \mathfrak{Y}'_\mu)\big)^-\,\big] \tag{34}$$

and in $A_*OQ_d(p'')$, the cycle class $\big[\tau(p'')^{-1}\big(W_\lambda(p) \cap W'_\mu(p)\big)\big]_k$ is equal to the right-hand side of (34).

Proof. By a dimension count which uses Proposition 2, the irreducible components of dimension k in $W_\lambda(p) \cap W'_\mu(p)$ are the ones indicated on the right-hand side of (34). As in [KT2], now, the result follows from the rational equivalence $\{p\} \sim \{p'\}$ on $\mathbb{P}^1$, pulled back to $Y := (\mathbb{P}^1 \times W_\lambda(p)) \cap W'_\mu$ (or further pulled back to $OQ_d(p'')$), once we know that the irreducible components of $W_\lambda(p) \cap W'_\mu(p)$ of dimension k are generically smooth and in the closure of the complement of the fiber of Y over p (and that this remains true after pullback by $\tau(p'')$). The 'in the closure' portion of the claim follows by an argument involving the Kontsevich compactification of OM_d, as in op. cit. Generic smoothness is clear for $(\mathrm{ev}^p)^{-1}(\mathfrak{X}_\lambda \cap \mathfrak{X}'_\mu)$. Transverality of a general translate also establishes generic smoothness for the other component, once we notice that any point x in a dense open subset of $\beta_1((\mathrm{ev}_1^p)^{-1}(\mathfrak{Y}_\lambda \cap \mathfrak{Y}'_\mu))$ has the property that for any local $\mathbb{C}$-algebra R with residue field $R/\mathfrak{m} \simeq \mathbb{C}$ and any $\psi\colon R \to W_\lambda(p) \cap W'_\mu(p)$ with closed point mapping to x, the map ψ factors through the restriction of β_1 to $\pi_1^{-1}(\{p\} \times OM_{d-1})$.

This assertion follows from elementary linear algebra, but because of some tricky cases involving parity, we give a sketch of the argument. Fix a basis $\{v_i\}$ of V so that the symmetric form is given by $\langle v_i, v_j\rangle = \delta_{i+j,2n+3}$. Without loss of generality, the two general-position flags are

$$F_i = \mathrm{Span}(v_1, \ldots, v_i)$$

and

$$G_i^{(0)} = \mathrm{Span}(v_{2n+3-i}, \ldots, v_{2n+2}),$$

where the latter specifies G_{n+1} or $\widetilde{G}_{n+1}$ equal to $\mathrm{Span}(v_{n+2}, \ldots, v_{2n+2})$ according to parity; see (22). We will show that the condition on x holds whenever x is in the preimage of the intersection of the Schubert *cells* corresponding to $\mathfrak{Y}_\lambda$ and $\mathfrak{Y}'_\mu$, subject to the further condition that the line on OG parametrized by the point in $OG(n-1,2n+2)$ is incident to $\mathfrak{X}_\lambda$ and $\mathfrak{X}'_\mu$ at two *distinct* points.

Consider first the case $\ell(\lambda)=1$. Let x correspond to $(n-1)$-dimensional $A\subset V$ at the point p. The condition to be in the Schubert cell for $\mathfrak{Y}_\lambda$ implies that $A\cap F_n^\perp=0$, so $\mathrm{rk}(A\to V/F^{(i)}_{n+1})=n-1$ for any i. By Definition 4, the sheaf sequence corresponding to ψ satisfies the rank condition

$$\mathrm{rk}(\mathcal{E}\to\mathcal{O}\otimes V/F^{(0)}_{n+1})\leqslant n-1. \tag{35}$$

Turning to the conditions coming from μ, we have $\mathrm{rk}(A\cap G^{(1)}_{n+1})=n-\ell$, from membership in the Schubert cell. Suppose n is even, so that $F^{(0)}_{n+1}=\widetilde{F}_{n+1}$ and $G^{(1)}=G_{n+1}$ are disjoint. Note that in this case Definition 4 imposes the condition

$$\mathrm{rk}(\mathcal{E}\to\mathcal{O}\otimes V/G_{n+1})\leqslant n-\ell. \tag{36}$$

The following basic argument is used to show that ψ factors through the restriction of β_1 to $\pi_1^{-1}(\{p\}\times OM_{d-1})$. We have a sheaf sequence on $\mathbb{P}^1_R$; after restricting to $\mathbb{A}^1_R$ the sheaf $\mathcal{E}$ can be trivialized, so let us assume the map to $\mathcal{O}\otimes V$ is given by the $(2n+2)\times(n+1)$ matrix L with values in $R[t]$, with coordinates assigned so the top half of the matrix corresponds to $\widetilde{F}_{n+1}$ and the bottom half corresponds to G_{n+1}. We may assume $t=0$ defines p, and also assume that mod $\mathfrak{m}$, the rightmost two columns of L vanish at $t=0$. We localize at $\mathfrak{m}+tR[t]$. It suffices to show that conditions (35) and (36) imply, after column operations, that the rightmost two columns of L have values in the ideal generated by t. We have $\mathrm{rk}(A\to V/F_{n+1})=n-1$, that is, some $(n-1)\times(n-1)$ minor in the bottom half of L has full rank. Now by performing column operations and invoking (35) we have all the entries in the bottom right $(n+1)\times 2$ submatrix of L lying in the ideal (t). Let L' denote the top right $(n+1)\times 2$ submatrix of L. The remaining isotropicity and rank conditions amount to $UL'=0 \bmod t$ for some matrix U, whose entries are polynomial functions of the entries of L in the first $n-1$ columns. The condition that the line corresponding to A meets the Schubert varieties in distinct points implies that the nullspace of U is trivial, and hence L' has entries in (t) as well.

If, instead, n is odd, we use the fact that $\mathrm{rk}(A\cap G_{n+1})=n+1-\ell$ (also a condition to be in the Schubert cell). From Definition 4,

$$\mathrm{rk}(\mathcal{E}\to\mathcal{O}\otimes V/G_{n+1})\leqslant \mathrm{rk}(\mathcal{E}\to\mathcal{O}\otimes V/G_n^\perp)\leqslant n+1-\ell. \tag{37}$$

Now $F^{(0)}_{n+1}=F_{n+1}$ and G_{n+1} are disjoint, and the basic argument applies, using (35) and (37).

In case $\ell(\lambda)=2$, we have $A\cap F^{(0)}_{n+1}=0$ and (35) still holds, so the argument is the same. □

We now establish the rational equivalences on OQ_d — and on $OQ_d(p'')$ — which directly imply the quantum Giambelli formula of Theorem 1.

Proposition 5. *Fix $\lambda\in\mathcal{D}_n$ with $\ell=\ell(\lambda)\geqslant 3$. Set $r=2\lfloor(\ell+1)/2\rfloor$. Let p, p', p'' denote distinct points in $\mathbb{P}^1$. Then we have the following identity of cycle classes*

$$\big[\,(\mathrm{ev}^p)^{-1}(\mathfrak{X}_\lambda)^-\big]=\sum_{j=1}^{r-1}(-1)^{j-1}\big[\,\big((\mathrm{ev}^p)^{-1}(\mathfrak{X}_{\lambda_j,\lambda_r})\cap(\mathrm{ev}^{p'})^{-1}(\mathfrak{X}'_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}})\big)^-\big], \tag{38}$$

both on OQ_d and on $OQ_d(p'')$, where $\mathfrak{X}'_\mu$ denotes the translate of $\mathfrak{X}_\mu$ by a generally chosen element of the group SO_{2n+2}.

Proof. Combining parts (a) and (b) of Proposition 4 gives

$$\begin{aligned}&\big[\,\big((\mathrm{ev}^p)^{-1}(\mathfrak{X}_{\lambda_j,\lambda_r}) \cap (\mathrm{ev}^{p'})^{-1}(\mathfrak{X}'_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}})\big)^{-}\,\big]\\ &\quad= \big[\,(\mathrm{ev}^p)^{-1}(\mathfrak{X}_{\lambda_j,\lambda_r} \cap \mathfrak{X}'_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}})^{-}\,\big] + \big[\,\beta_1\big((\mathrm{ev}_1^p)^{-1}(\mathfrak{Y}_{\lambda_j,\lambda_r} \cap \mathfrak{Y}'_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}})\big)^{-}\,\big]\end{aligned}$$

for each j, with $1 \leqslant j \leqslant r-1$. Now (38) follows by summing and applying (i) and (ii) of Corollary 4. ☐

Theorem 4. *Suppose $\lambda \in \mathcal{D}_n$, with $\ell = \ell(\lambda) \geqslant 3$, and set $r = 2\lfloor(\ell+1)/2\rfloor$. Then we have the following identity in $QH^*(OG)$:*

$$\tau_\lambda = \sum_{j=1}^{r-1} (-1)^{j-1} \tau_{\lambda_j,\lambda_r} \tau_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}}. \tag{39}$$

Proof. The classical component of (39) follows from the classical Giambelli formula for OG. To handle the remaining terms, apply a refined cap product operation [F, §8.1] along $\mathrm{ev}(p'')$ to general translates of $\mathfrak{X}_\mu$ for all $\mu \in \mathcal{D}_n$ with $|\mu| = h(n,d) - |\lambda|$, and invoke Corollaries 3 and 2 (as in the proof of [KT2, Thm. 5]). ☐

6. Quantum Schubert calculus

Our aim in this Section is to use Theorem 1 and the algebra of $\widetilde{P}$-polynomials to find combinatorial rules that compute some of the quantum structure constants that appear in the quantum product of two Schubert classes.

6.1. **Algebraic background.** Let $\mathcal{E}_n$ denote the set of all partitions λ with $\lambda_1 \leqslant n$. The main properties of $\widetilde{Q}$-polynomials that we need are collected in [KT2, §2.1 and §6.1]. They imply corresponding facts about the $\widetilde{P}$-polynomials, in particular, that the set $\{\widetilde{P}_\lambda(X) \mid \lambda \in \mathcal{E}_n\}$ is a free $\mathbb{Z}$-basis of the ring Λ'_n that they span. Hence, there exist integers $f(\lambda,\mu;\,\nu)$ such that

$$\widetilde{P}_\lambda(X)\,\widetilde{P}_\mu(X) = \sum_\nu f(\lambda,\mu;\,\nu)\,\widetilde{P}_\nu(X); \tag{40}$$

the constants $f(\lambda,\mu;\,\nu)$ are independent of n, and defined for any $\lambda,\mu,\nu \in \mathcal{E}_n$. The corresponding coefficients $e(\lambda,\mu;\,\nu)$ in the expansion of the product $\widetilde{Q}_\lambda(X)\,\widetilde{Q}_\mu(X)$ are related to these by the equation

$$e(\lambda,\mu;\,\nu) = 2^{\ell(\lambda)+\ell(\mu)-\ell(\nu)} f(\lambda,\mu;\,\nu). \tag{41}$$

There are explicit combinatorial rules (involving signs in general) for computing the integers $f(\lambda,\mu;\,\nu)$, which follow from corresponding formulas for decomposing products of Hall-Littlewood polynomials; for more details, see [KT2, §6.1]. Define the connected components of a skew Young diagram by specifying that two boxes are connected if they share a vertex or an edge. We then have the following Pieri type formula for λ *strict*:

$$\widetilde{P}_\lambda(X)\,\widetilde{P}_k(X) = \sum_\mu 2^{N'(\lambda,\mu)}\,\widetilde{P}_\mu(X), \tag{42}$$

where the sum is over all partitions $\mu \supset \lambda$ with $|\mu| = |\lambda| + k$ such that μ/λ is a horizontal strip, and $N'(\lambda, \mu)$ is one less than the number of connected components of μ/λ. In particular, we have $\widetilde{P}_\lambda(X)\widetilde{P}_n(X) = \widetilde{P}_{(n,\lambda)}(X)$ for all $\lambda \in \mathcal{D}_n$.

When λ, μ and ν are strict partitions, the $f(\lambda, \mu;\, \nu)$ are classical structure constants for $OG(n+1, 2n+2)$,

$$\tau_\lambda \tau_\mu = \sum_{\nu \in \mathcal{D}_n} f(\lambda, \mu;\, \nu)\, \tau_\nu,$$

and hence are nonnegative integers. In this case, Stembridge [St] has given a combinatorial rule for the numbers $f(\lambda, \mu;\, \nu)$, analogous to the usual Littlewood-Richardson rule in type A. Specifically, $f(\lambda, \mu;\, \nu)$ is equal to the number of marked tableaux of weight λ on the shifted skew shape $\mathcal{S}(\nu/\mu)$ satisfying certain conditions (see [St] and [P, Sect. 6] for more details).

6.2. **Quantum multiplication.** Recall from the Introduction that for any $\lambda, \mu \in \mathcal{D}_n$ there is a formula

$$\tau_\lambda \cdot \tau_\mu = \sum f_{\lambda\mu}^{\nu}(n)\, \tau_\nu\, q^d$$

in $QH^*(OG(n+1, 2n+2))$, with each $f_{\lambda\mu}^{\nu}(n)$ equal to a Gromov–Witten invariant $\langle \tau_\lambda, \tau_\mu, \tau_{\widehat{\nu}} \rangle_d$ (defined when $|\lambda| + |\mu| = |\nu| + 2nd$). The nonnegative integer $f_{\lambda\mu}^{\nu}(n)$ counts the number of degree-d rational maps $\psi : \mathbb{P}^1 \to OG$ such that $\psi(0) \in \mathfrak{X}_\lambda$, $\psi(1) \in \mathfrak{X}_\mu$ and $\psi(\infty) \in \mathfrak{X}_{\widehat{\nu}}$, when the three Schubert varieties $\mathfrak{X}_\lambda$, $\mathfrak{X}_\mu$ and $\mathfrak{X}_{\widehat{\nu}}$ are in general position.

We adopt the convention that $\tau_\lambda = 0$ for all non-strict partitions λ. Now Theorem 1 and the Pieri rule (42) give

Corollary 5 (Quantum Pieri Rule). *For any $\lambda \in \mathcal{D}_n$ and $k \geqslant 0$ we have*

$$\tau_\lambda \tau_k = \sum_{\mu} 2^{N'(\lambda,\mu)} \tau_\mu + \sum_{\mu \supset (n,n)} 2^{N'(\lambda,\mu)} \tau_{\mu \smallsetminus (n,n)}\, q$$

where both sums are over $\mu \supset \lambda$ with $|\mu| = |\lambda| + k$ such that μ/λ is a horizontal strip, and the second sum is restricted to those μ with two parts equal to n.

In recent work with Buch [BKT], we give a more direct proof of the quantum Pieri rule for OG, and the corresponding rule for the Lagrangian Grassmannian.

For any $d, n \geqslant 0$ and partition ν, let (n^d, ν) denote the partition

$$(n, n, \ldots, n, \nu_1, \nu_2, \ldots),$$

where n appears d times before the first component ν_1 of ν. Theorem 1 now gives

Theorem 5. *For any $d \geqslant 0$ and strict partitions $\lambda, \mu, \nu \in \mathcal{D}_n$ with $|\nu| = |\lambda| + |\mu| - 2nd$, the quantum structure constant $f_{\lambda\mu}^{\nu}(n)$ satisfies $f_{\lambda\mu}^{\nu}(n) = f(\lambda, \mu;\, (n^{2d}, \nu))$.*

We deduce that for any strict partitions $\lambda, \mu, \nu \in \mathcal{D}_n$, the coefficient $f(\lambda, \mu;\, (n^d, \nu))$ is a nonnegative integer. The constants $f(\lambda, \mu;\, \nu)$ can be negative; for example

$$f(\rho_3, \rho_3;\, (4, 4, 2, 2)) = -1.$$

This follows from the Remark in [KT2, §6.2].

6.3. **The relation to $QH^*(LG(n-1,2n-2))$.** The quantum Pieri rule of Proposition 5 implies that

$$\tau_n\,\tau_\lambda = \begin{cases} \tau_{(n,\lambda)} & \text{if } \lambda_1 < n, \\ \tau_{\lambda\smallsetminus(n)}\, q & \text{if } \lambda_1 = n \end{cases}$$

in the quantum cohomology ring of $OG(n+1,2n+2)$. Therefore, to compute all the Gromov–Witten invariants for OG, it suffices to evaluate the $\langle \tau_\lambda, \tau_\mu, \tau_\nu\rangle_d$ for $\mu,\nu \in \mathcal{D}_{n-1}$. Define a map $*: \mathcal{D}_n \to \mathcal{D}_{n-1}$ by setting $\lambda^* = (n-\lambda_\ell, \ldots, n-\lambda_1)$ for any partition λ of length ℓ, and $(0)^* = (0)$.

Partitions in $\mathcal{D}_{n-1}$ also parametrize the Schubert classes σ_λ in the (quantum) cohomology ring of the Lagrangian Grassmannian $LG(n-1,2n-2)$, which was studied in [KT2]. For the remainder of this paper, we let $': \mathcal{D}_{n-1} \to \mathcal{D}_{n-1}$ denote the duality involution for this space, so that the parts of λ' complement the parts of λ in the set $\{1,2,\ldots,n-1\}$. Notice that the restriction of $*$ to $\mathcal{D}_{n-1}$ defines a second involution on this set, which was considered in [KT2, §6.3].

Theorem 6. *Suppose that $\lambda \in \mathcal{D}_n$ is a non-zero partition with $\ell(\lambda) = 2d+e+1$ for some nonnegative integers d and e. For any $\mu,\nu \in \mathcal{D}_{n-1}$, we have an equality*

$$\langle \tau_\lambda, \tau_\mu, \tau_\nu\rangle_d = \langle \sigma_{\lambda^*}, \sigma_{\mu'}, \sigma_{\nu'}\rangle_e \tag{43}$$

of Gromov–Witten invariants for $OG(n+1,2n+2)$ and $LG(n-1,2n-2)$, respectively. If λ is zero or $\ell(\lambda) < 2d+1$, then $\langle \tau_\lambda, \tau_\mu, \tau_\nu\rangle_d = 0$.

Proof. Assume first that $\lambda_1 < n$, so $\lambda \in \mathcal{D}_{n-1}$. We then have

$$\begin{aligned} \langle \tau_\lambda, \tau_\mu, \tau_\nu\rangle_d &= f(\lambda,\mu;\, (n^{2d+1},\nu')) \\ &= 2^{n+2d-\ell(\lambda)-\ell(\mu)-\ell(\nu)}\, e(\lambda,\mu;\, (n^{2d+1},\nu')) \\ &= 2^{n+4d+1-\ell(\lambda)-\ell(\mu)-\ell(\nu)}\, \langle \sigma_\lambda, \sigma_\mu, \sigma_\nu\rangle_{2d+1} \end{aligned}$$

where the last equality comes from [KT2, Thm. 6]. The result now follows by applying the eight-fold symmetry [KT2, Thm. 7] for $QH^*(LG(n-1,2n-2))$, which dictates

$$2^{n+2d}\,\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu\rangle_{2d+1} = 2^{\ell(\mu)+\ell(\nu)+e}\,\langle \sigma_{\lambda^*}, \sigma_{\mu'}, \sigma_{\nu'}\rangle_e. \tag{44}$$

If $\lambda_1 = n$, then

$$\langle \tau_\lambda, \tau_\mu, \tau_\nu\rangle_d = \langle \tau_{\lambda\smallsetminus(n)}, \tau_\mu, \tau_{(n,\nu)}\rangle_d = f(\lambda\smallsetminus(n),\mu;\, (n^{2d},\nu')),$$

and the previous analysis applies, since $\lambda^* = (\lambda \smallsetminus (n))^*$. □

Of course this theorem also provides an equality of Gromov–Witten invariants going the other way. For any $\lambda,\mu,\nu \in \mathcal{D}_{n-1}$, we have

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu\rangle_e = \begin{cases} \langle \tau_{\lambda^*}, \tau_{\mu'}, \tau_{\nu'}\rangle_d & \text{if } \ell(\lambda)-e = 2d+1 \text{ is odd}, \\ \langle \tau_{(n,\lambda^*)}, \tau_{\mu'}, \tau_{\nu'}\rangle_d & \text{if } \ell(\lambda)-e = 2d \text{ is even}. \end{cases}$$

The $(\mathbb{Z}/2\mathbb{Z})^3$-symmetry (44) enjoyed by the Gromov–Witten invariants for $LG(n-1,2n-2)$ implies a similar one for $QH^*(OG)$.

Proposition 6. *Let $\lambda \in \mathcal{D}_n$ be non-zero and $\mu,\nu \in \mathcal{D}_{n-1}$. For any $d,e \geqslant 0$ with $2d+e+1 = \ell(\lambda)$, we have*

$$2^{\ell(\mu)+\ell(\nu)+e+\delta}\,\langle \tau_\lambda, \tau_\mu, \tau_\nu\rangle_d = 2^{n+2d} \begin{cases} \langle \tau_{\lambda^*}, \tau_{\mu'}, \tau_{\nu'}\rangle_g & \textit{if } e = 2g+1 \textit{ is odd}, \\ \langle \tau_{(n,\lambda^*)}, \tau_{\mu'}, \tau_{\nu'}\rangle_g & \textit{if } e = 2g \textit{ is even}, \end{cases}$$

where $\delta = \delta_{\lambda_1,n}$ is the Kronecker symbol.

We now obtain orthogonal analogues of [KT2, Prop. 10] and [KT2, Cor. 8].

Corollary 6. *Let λ, μ, ν and δ be as in Proposition 6. Then the inequalities*

$$\ell(\mu) + \ell(\nu) - n + \delta \leqslant 2d \leqslant \ell(\lambda) + \ell(\mu) + \ell(\nu) - n \tag{45}$$

are necessary conditions for the Gromov–Witten invariant $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d$ to be nonzero. Moreover, if the two sides of either of the inequalities in (45) differ by 0 or 1, then $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d$ is related by the eight-fold symmetry to a classical structure constant.

Corollary 7. *For any $\lambda \in \mathcal{D}_n$, we have*

$$\tau_\lambda \cdot \tau_{\rho_{n-1}} = \begin{cases} \tau_{\lambda^{*\prime}}\, q^d & \text{if } \ell(\lambda) = 2d \text{ is even,} \\ \tau_{(n,\lambda^{*\prime})}\, q^d & \text{if } \ell(\lambda) = 2d+1 \text{ is odd.} \end{cases}$$

in $QH^*(OG)$. In particular,

$$\tau_{\rho_n} \cdot \tau_{\rho_n} = \begin{cases} \tau_n\, q^{n/2} & \text{if } n \text{ is even,} \\ q^{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

7. Appendix: An identity in $\widetilde{P}$-polynomials

We give a proof of the following identity, which is used to simplify a formula for degeneracy loci in type D [KT1]. The proof uses the algebraic formalism of §2.2.

Proposition 7. *Let $X = (x_1, \ldots, x_n)$ be an n-tuple of variables, and consider also $\widetilde{X} = (-x_1, x_2, \ldots, x_n)$ and $X' = (x_2, \ldots, x_n)$. Then, for any $\lambda \in \mathcal{E}_n$ of length $\ell \geqslant 1$ we have*

$$\sum_{i=1}^{\ell} (-1)^{i-1} \widetilde{P}_{\lambda \smallsetminus \{\lambda_i\}}(X) e_{\lambda_i}(X') = \widetilde{P}_\lambda(\widetilde{X}) + (-1)^{\ell+1} \widetilde{P}_\lambda(X). \tag{46}$$

Proof. By homogeneity, (46) is equivalent to the identity

$$\sum_{i=1}^{\ell} (-1)^{i-1} \widetilde{Q}_{\lambda \smallsetminus \{\lambda_i\}}(X) \widetilde{Q}_{\lambda_i}(X') = \frac{1}{2}(\widetilde{Q}_\lambda(\widetilde{X}) + (-1)^{\ell+1} \widetilde{Q}_\lambda(X)). \tag{47}$$

To establish (47), we use identity (11) and are reduced to

$$\sum_{i=1}^{\ell} (-1)^{i-1} \widetilde{Q}_{\lambda_i}(X') \sum_{\mu \in B(\lambda \smallsetminus \{\lambda_i\}, k)} \widetilde{Q}_\mu(X') = \begin{cases} \sum_{\mu \in B(\lambda,k)} \widetilde{Q}_\mu(X'), & \text{if } k \neq \ell \bmod 2, \\ 0 & \text{if } k = \ell \bmod 2, \end{cases}$$

for all integers k, where $B(\lambda, k)$ is defined as in the proof of Proposition 1. This corresponds to an identity in the algebra $\mathcal{A}$ of formal variables with imposed relations of [KT2, §2.3], which is similar to the algebra $\mathcal{B}$ of §2.2, except that only single bars appear.

Using the equalities

$$[a,b](c) - [a,c](b) + [b,c](a) = 0 \tag{48}$$

and

$$[a,b](\overline{c}) - [a,c](\overline{b}) + [b,c](\overline{a}) = 0 \tag{49}$$

in $\mathcal{A}$, one can verify, for each combination of parities of k and ℓ, that the corresponding identity in $\mathcal{A}$ is true (one case, that of k odd, ℓ even, uses also the identity (17)). For example, when k is even and ℓ is odd, we need to show that

$$\sum_{i=1}^{\ell}(-1)^{i-1}(\lambda_i)\sum_{\mu\in B(\lambda\smallsetminus\{\lambda_i\},k)}\sum\epsilon(\mu,\nu)(\nu_1,\nu_2)\cdots(\nu_{\ell-2},\nu_{\ell-1})=\sum_{\nu\in B(\lambda,k)}(\nu) \tag{50}$$

where the innermost sum on the left is over all $(\ell-2)(\ell-4)\cdots(1)$ ways to write the set of entries of μ as a union of pairs $\{\nu_1,\nu_2\}\cup\cdots\cup\{\nu_{\ell-2},\nu_{\ell-1}\}$. Using (48), the sum of the terms on the left hand side which contain a pair with exactly one bar vanishes. The remaining terms are seen, using (48) and (49), to be equal to the Pfaffian expansion of the right-hand side of (50). □

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