Schubert Calculus on the Arithmetic Grassmannian

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Abstract

Let $\overline{G}$ be the arithmetic Grassmannian over Spec$\mathbb{Z}$ with the natural invariant Kähler metric on $G(\mathbb{C})$. We study the combinatorics of the arithmetic Schubert calculus in the Arakelov Chow ring $CH(\overline{G})$. We obtain formulas for the arithmetic Littlewood-Richardson numbers and the Faltings height of G under the Plücker embedding, using 'rim hook operations' on Young diagrams. An analysis of the duality involution leads to new combinatorial relations among Kostka numbers.

1 Introduction

Arakelov geometry provides a method of measuring the complexity of a system of diophantine equations. Such a system defines an arithmetic variety in projective space, which is then studied using techniques of intersection theory and hermitian complex geometry. The arithmetic complexity of this variety is controlled by numerical invariants called *heights*. Although their exact computation is difficult, often a good bound for these numbers is enough to prove finiteness results.

The modern theory, developed by Gillet and Soulé [GS1], attaches to each arithmetic variety X a large ring, the *arithmetic Chow ring*. Following Faltings [F], the height of X is defined as its arithmetic degree with respect to the canonical hermitian line bundle, in analogy with the geometric notion of degree. More generally, one expects that all concepts and results from geometric intersection theory should have analogues over the integers (cf. [S]).

There are very few examples where explicit formulas for heights are known; their calculation is often equivalent to evaluating intricate fiber integrals. For varieties whose complexifications are hermitian symmetric spaces, such as Grassmannians, a smaller *Arakelov Chow ring* is available, which is a subring of the larger one. In this case there are more computational tools at hand: one reduces the problem to a calculation of secondary characteristic classes called *Bott-Chern forms*. These forms are objects of pure complex geometry, and are defined with no reference to arithmetic at all.

Products in the Arakelov Chow ring of projective space were computed in the foundational work of Gillet and Soulé [GS2]. A corresponding analysis for Grassmannians was done by Maillot [Ma]; he formulated an 'arithmetic Schubert calculus' analogous to the classical one. The combinatorial formulas obtained in [Ma], although explicit, were quite complicated. In this article we arrive at a simpler picture.

Let $\overline{G} = (G, \omega_G)$ denote the arithmetic Grassmannian $G = G(m, n)$ parametrizing m-planes in $(m+n)$-space (over any field), with the natural invariant Kähler form ω_G on $G(\mathbb{C})$. We present formulas for the arithmetic intersections of the classes of Schubert varieties in the Arakelov Chow ring $CH(\overline{G})$. More precisely, if $\overline{Q}$ is the universal quotient bundle on G with the induced invariant metric, there are arithmetic Schubert classes $\widehat{s}_\lambda(\overline{Q})$ in $CH(\overline{G})$, one for each Young diagram λ contained in an $n \times m$ rectangle (m^n). Such λ also correspond to classes of ω_G-harmonic differential forms $s_\lambda(\overline{Q})$ in the same ring. The multiplication rule

$$\widehat{s}_\lambda(\overline{Q})\widehat{s}_\mu(\overline{Q}) = \sum_{\nu \subset (m^n)} N_{\lambda\mu}^\nu \widehat{s}_\nu(\overline{Q}) + \sum_{\nu \subset (m^n)} \widetilde{N}_{\lambda\mu}^\nu(m) s_\nu(\overline{Q})$$

with $N_{\lambda\mu}^\nu$ the classical Littlewood-Richardson numbers defines the *arithmetic Littlewood-Richardson numbers* $\widetilde{N}_{\lambda\mu}^\nu(m)$. Our aim is to obtain as explicit a description as possible for these numbers. We shall see in §4.1 that they depend on m but are independent of n.

We find the combinatorics of the arithmetic Schubert calculus quite fascinating. It is a 'deformation' of the classical theory where one encounters harmonic numbers, Littlewood-Richardson coefficients and signs. Our formula for the arithmetic Littlewood-Richardson numbers involves an operation on Young diagrams given by subtracting and then adding *rim hooks*. We apply our algorithm to calculate the Faltings height of the Grassmannian G in its Plücker embedding in projective space. This height was calculated by Maillot [Ma]; our formula is an improvement of his. For instance we obtain the following closed formula for the height of the Grassmannian $G(2,n)$ of 2-planes:

$$ht(G(2,n)) = \left(\mathcal{H}_{n+2} - \frac{2n+1}{2n+2}\right)\binom{2n+1}{n} - \frac{4^n}{n+1}.$$

Here $\mathcal{H}_k = \sum_{i=1}^{k} \frac{1}{i}$ is a harmonic number.

The ingredients for doing these calculations come from two different directions: complex differential geometry and combinatorics. In [T2] techniques for calculating Bott-Chern forms were developed with this problem in mind. Although they led to a new presentation of the Arakelov Chow ring, there were still combinatorial difficulties to resolve for applications to arithmetic Schubert calculus.

In the author's University of Chicago thesis [T1] an arithmetic Schubert calculus is established for the arithmetic Chow ring of any partial flag variety. The analysis of this more general situation requires a different method than that of [Ma]. Although the specialization of the general Schubert calculus of [T1] to the Grassmannian case is essentially identical to the one in [Ma], the change of approach is important.

Just as the Schur polynomials are essential tools in this work, the Schubert polynomials of Lascoux and Schutzenberger [LS] were needed to study arithmetic flag varieties. The decisive role of Schubert polynomials in this story stems from their use in the description of degeneracy loci (see [Fu2]), and provides an illustration of the well-known parallel between the arithmetic and geometric cases. The author has pursued this analogy further in [T3]. One crucial combinatorial property of Schubert polynomials, which we call the *ideal property*, is the reason they are useful in arithmetic geometry (cf. §2). Since Schubert polynomials specialize to Schur functions, the latter enjoy this property as well.

The missing combinatorial ingredient to simplify the arithmetic story for Grassmannians was a formula establishing the ideal property for Schur functions directly. Such a formula was shown to the author by Lascoux during the April 1997 Oberwolfach conference on Schubert varieties. The power of this formula lies in its utility for studying calculus in deformations of the cohomology ring of G. The same methods may be applied to study the Schubert calculus in the (small) quantum cohomology ring $QH^*(G(\mathbb{C}))$, obtaining some of the results of [BCF]. It is interesting to compare our work with [BCF]; the formulas and combinatorial phenomena in the two articles are strikingly similar.

The effect of the canonical duality isomorphism

$$CH(\overline{G}(m,n)) \cong CH(\overline{G}(n,m))$$

on our algorithm leads to non-trivial combinatorial identities. For instance the aforementioned height formula comes from our analysis of $G(n,2)$ rather than $G(2,n)$, a direct computation for the latter being much more involved. We give an algebraic proof of duality for a broader class of rings and discuss some combinatorial consequences.

Here is a brief outline of this article. Section 2 provides some combinatorial background on Young diagrams and symmetric functions. The ideal property of Schur functions is stated and proved. In §3 we introduce the Arakelov Chow ring $CH(\overline{G})$ and summarize the facts we need from previous work. This is the most 'arithmetic' part of the paper. We note however that all our arguments are algebraic and combinatorial; Arakelov theory is used only for motivation. The arithmetic Schubert calculus is the subject of §4. We give formulas for the arithmetic Littlewood-Richardson numbers in terms of the classical ones and 'rim hook operations'. This analysis is used in §5 to compute the Faltings height of the Grassmannian under its Plücker embedding. More complicated expressions for both these invariants were given in [Ma]; we compare the two approaches using duality. In Section 6 the duality isomorphism is investigated in a more general setting by algebraic methods. As a consequence we get some non-trivial combinatorial identities involving the Kostka numbers.

It is a pleasure to thank Alain Lascoux for stimulating discussions in the woods surrounding Oberwolfach and in particular for the formula establishing the ideal property of Schur polynomials. The author has benefitted much from conversations with William Fulton; he thanks him for encouragement and mathematical guidance.

2 Young diagrams and symmetric functions

In this section we give a brief description of the combinatorial notions that are relevant for the rest of the paper. Our main reference for this material is the book of Macdonald [M2]; we will mostly adopt the notational conventions there. For connections with geometry, see [Fu3].

We will identify a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ with its Young diagram of boxes; the conjugate partition λ' is the partition whose diagram is the transpose of λ. The number of (non-zero) parts of λ is the *length* of λ, denoted $l(\lambda)$. The inclusion relation $\mu \subset \lambda$ of partitions is defined by the containment of diagrams; in this case λ/μ denotes the corresponding skew diagram. The number of boxes in λ/μ is the *weight* of λ/μ, denoted $|\lambda/\mu|$. Thus λ is a partition of the number $|\lambda|$. For two partitions $\lambda = (\lambda_i)_{i\geqslant 0}$ and $\mu = (\mu_i)_{i\geqslant 0}$ we have the sum $\lambda + \mu = (\lambda_i + \mu_i)_{i\geqslant 0}$ and the set-theoretic difference $\lambda \smallsetminus \mu$.

Given a diagram λ and a box $x \in \lambda$, the *hook* H_x is the set of all boxes directly to the right and below x, including x itself. The corresponding *rim hook* R_x is the skew diagram obtained by projecting H_x along diagonals onto the boundary of λ. This is illustrated in Figure 1.

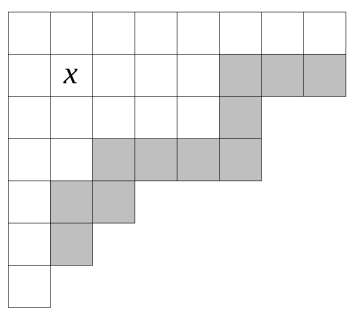

Figure 1: The rim hook corresponding to x

The number h_x of boxes in H_x (and R_x) is called the *length* of the hook (rim hook). We refer to a rim hook of length q as a *rim q-hook*. The *height* of R_x, denoted $ht(R_x)$, is one less than the number of rows it occupies.

Throughout this article we will use concise notation for collections of commuting variables, or *alphabets*. If $X = (X_1, \ldots, X_n)$ is a set of n indeterminants, we denote by $\Lambda_n := \mathbb{Z}[X]^{S_n}$ the ring of symmetric polynomials in X. There are many different bases for Λ_n, among which the most natural (and least obvious) is the basis of *Schur functions* $\{s_\lambda(X)\}$ for all partitions λ.

More generally, the *skew Schur functions* $s_{\lambda/\mu}$ are defined as follows: if $\lambda = k$ is a single positive integer then $s_k = h_k$ is the sum of all distinct monomials of degree k, while $s_0 = 1$ and $s_k = 0$ when $k < 0$. For two partitions λ and μ,

$$s_{\lambda/\mu} = \det(s_{\lambda_i - \mu_j - i + j})_{1 \leqslant i,j \leqslant l(\lambda)}.$$

This also defines $s_\lambda = s_{\lambda/\emptyset}$. Note that $s_{\lambda/\mu} = 0$ unless $\mu \subset \lambda$.

For $\Lambda_n \otimes_{\mathbb{Z}} \mathbb{Q}$ one has the $\mathbb{Q}$-basis of *power sums* $\{p_\lambda(X)\}$, again indexed by partitions. There is a unique inner product $\langle\ ,\ \rangle$ on Λ_n such that the Schur functions s_λ form an orthonormal set. The power sums p_λ are pairwise orthogonal for this same product.

Given another alphabet $Y = (Y_1, \ldots, Y_m)$, we can consider Schur functions $s_\lambda(Y)$, $s_\lambda(X, Y)$ in the Y variables and in the X and Y variables together. We let (m^n) denote the partition $(m, \ldots, m)$ of weight mn. The formula referred to in the introduction is expressed in

Proposition 1 $s_\lambda(X) = \sum_{\mu \subset (m^n)} (-1)^{|\mu|} s_{\lambda/\mu}(X, Y) s_{\mu'}(Y).$

Proof. Given r alphabets $Z_1, \ldots, Z_r$ and an r-tuple ν of integers, define the *multi-Schur function*

$$s_\nu(Z_1, \ldots, Z_r) = \det(s_{\nu_i + j - i}(Z_i))_{1 \leqslant i,j \leqslant r}$$

as in [M1] (3.1′). For $\alpha \in \mathbb{Z}^m$ and $\beta \in \mathbb{Z}^n$, $s_{\alpha,\beta}(D, D')$ denotes the multi-Schur function indexed by the concatenation of α and β and alphabets

$$Z_1 = \cdots = Z_m = D \qquad \text{and} \qquad Z_{m+1} = \cdots = Z_{m+n} = D'.$$

Using [M1] (3.4) we obtain the determinant factorization

$$s_{\alpha,\beta}(D, D') = s_\alpha(D) s_\beta(D' - D)$$

for any alphabet D of cardinality at most m; here $D' - D$ is the formal difference of alphabets. It now follows that

$$s_\lambda(X) = s_{(0^m),\lambda}(Y, X + Y)$$

for any partition λ, where (0^m) denotes a sequence of m zeroes. The sum in the proposition is the Laplace expansion of the determinant

$$s_{(0^m),\lambda}(Y, X + Y)$$

along the subfamily of the first m rows. □

We now explain the connection between Proposition 1 and the ideal property of Schubert polynomials, which was used in the author's study of arithmetic flag varieties. Let $N = n+m$ and consider the ring $R_N = \mathbb{Z}[X,Y]/I_N$, where I_N is the ideal generated by the Schur polynomials $s_\lambda(X,Y)$ in both sets of variables for $\lambda \neq 0$. R_N can be identified with the Chow ring of the flag variety $Fl(N)$ parametrizing complete flags in N-space.

Let $S^{(N)}$ denote the set of permutations $w : \mathbb{N} \to \mathbb{N}$ that leave all but finitely many numbers fixed and have no descents after the first $N+1$ values. Note that the symmetric group S_N is naturally contained in $S^{(N)}$. For each $w \in S^{(N)}$ Lascoux and Schützenberger [LS] define a *Schubert polynomial* $\mathfrak{S}_w \in \mathbb{Z}[X,Y]$. The set $\{\mathfrak{S}_w \mid w \in S^{(N)}\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[X,Y]$, while $\{\mathfrak{S}_w \mid w \in S_N\}$ is a $\mathbb{Z}$-basis of the quotient ring R_N.

The *ideal property* of Schubert polynomials states that if $w \in S^{(N)} \smallsetminus S_N$, then $\mathfrak{S}_w$ is contained in the ideal I_N. For a simple proof, see Lemma 1 of [T1]. If v is a permutation such that $v(i) < v(i+1)$ when $i \neq n$ (i.e. a *Grassmannian* permutation), then $\mathfrak{S}_v = s_{\lambda_v}(X)$ is a Schur polynomial in the variables $X_1, \ldots, X_n$; here

$$\lambda_v = (v(n) - n, v(n-1) - (n-1), \ldots, v(1) - 1).$$

In case v is not in S_N, the equation of Proposition 1 provides a direct proof of the ideal property for $s_{\lambda_v}(X)$.

3 The Arakelov Chow ring $CH(\overline{G})$

In this section we will introduce the main object of study. We refer to the foundational papers of Gillet and Soulé [GS1] [GS2] and the expositions [SABK] [S] for background as well as [Ma] [T2] for previous work.

Let $G = G(m,n)$ denote the Grassmannian over Spec$\mathbb{Z}$. For any field k the set of points $G(k)$ parametrizes m-dimensional subspaces in k^N where $N = m+n$. G is a smooth arithmetic variety of absolute dimension $d = mn+1$. The complex manifold $G(\mathbb{C})$ is endowed with a natural $U(N)$-invariant metric coming from the Kähler form $\omega_G = c_1(\overline{Q(\mathbb{C})})$; we let $\overline{G} = (G, \omega_G)$. There are three rings attached to G: the Chow ring $CH(G)$, the ring $\mathcal{H}(G_\mathbb{R})$ of real ω_G-harmonic differential forms on $G(\mathbb{C})$, and the Arakelov Chow ring $CH(\overline{G})$. For the first two there are natural isomorphisms

$$CH(G) \otimes_\mathbb{Z} \mathbb{R} \cong \mathcal{H}(G_\mathbb{R}) \cong H^*(G(\mathbb{C}), \mathbb{R}),$$

the last ring being cohomology with real coefficients. The Arakelov Chow ring $CH(\overline{G})$ sits in a short exact sequence

$$0 \longrightarrow \mathcal{H}(G_{\mathbb{R}}) \xrightarrow{a} CH(\overline{G}) \xrightarrow{\zeta} CH(G) \longrightarrow 0. \tag{1}$$

Over G there is a universal exact sequence of vector bundles

$$\mathcal{E}: \ 0 \to S \to E \to Q \to 0. \tag{2}$$

We give the trivial bundle $E(\mathbb{C})$ the trivial hermitian metric and the tautological subbundle $S(\mathbb{C})$ and quotient bundle $Q(\mathbb{C})$ the induced metrics. (2) then becomes a sequence of *hermitian vector bundles*:

$$\overline{\mathcal{E}}: \ 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0. \tag{3}$$

For each symmetric polynomial ϕ there are characteristic forms and classes associated to these bundles. We have three different kinds: the usual classes $\phi(Q)$, $\phi(S)$ in $CH(G)$, the characteristic forms $\phi(\overline{Q})$, $\phi(\overline{S})$ in $\mathcal{H}(G_{\mathbb{R}})$ given by Chern-Weil theory, and the arithmetic classes $\widehat{\phi}(\overline{Q})$, $\widehat{\phi}(\overline{S})$ in $CH(\overline{G})$. We refer to these elements using three sets of formal 'root variables' $\{\mathrm{x}, \mathrm{y}\}$, $\{x, y\}$ and $\{\widehat{x}, \widehat{y}\}$, respectively. For instance, symmetric functions ϕ in the variables $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ denote the characteristic forms $\phi(\overline{Q})$ and $\phi(\overline{S})$ (which we also identify, via the inclusion a, with elements in $CH(\overline{G})$). The *harmonic numbers* $\mathcal{H}_k$ defined by

$$\mathcal{H}_0 = 0, \qquad \mathcal{H}_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$$

will play an important role in the description of $CH(\overline{G})$.

Let $H = S_n \times S_m$ be the product of two symmetric groups. There is a natural H-action on the polynomial ring $\mathbb{Z}[\mathrm{x}, \mathrm{y}]$ by permuting the two sets of variables. The following isomorphism is well known (cf. [Fu1] Ex. 14.6.6):

$$CH(G) \cong \frac{\mathbb{Z}[\mathrm{x}, \mathrm{y}]^H}{\langle e_k(\mathrm{x}, \mathrm{y}) = 0, \ k \geqslant 1 \rangle} \tag{4}$$

where $e_k(\mathrm{x}, \mathrm{y})$ is the k-th elementary symmetric polynomial in x and y. A presentation of $CH(\overline{G})$ is obtained by a deformation of this construction. Consider the polynomial rings

$$\mathcal{A} = \mathbb{Z}[\widehat{x}, \widehat{y}]^H \qquad \text{and} \qquad \mathcal{B} = \mathbb{R}[x, y]^H,$$

which are the rings $\mathbb{Z}[\widehat{c}(\overline{Q}),\widehat{c}(\overline{S})]$ and $\mathbb{R}[c(\overline{Q}),c(\overline{S})]$ in 'root notation'. The product $\widehat{\alpha}\cdot\beta=\alpha\beta$ that 'forgets the hats' turns $\mathcal{B}$ into an $\mathcal{A}$-module. In this situation the direct sum $\mathcal{A}\oplus\mathcal{B}$ inherits a natural ring structure for which $\mathcal{B}$ is a square zero ideal; we use $\cdot$ to denote the induced product on $\mathcal{A}\oplus\mathcal{B}$. By convention any product $\prod x_i y_j$ denotes $(0,\prod x_i y_j)$. The Arakelov Chow ring $CH(\overline{G})$ is isomorphic to the graded ring $(\mathcal{A}\oplus\mathcal{B},\cdot)$ modulo the two relations

$$e_k(x,y)=0 \tag{5}$$

and

$$e_k(\widehat{x},\widehat{y})=(-1)^{k-1}\mathcal{H}_{k-1}p_{k-1}(x) \tag{6}$$

for all $k\geqslant 1$; here $p_k(x)=p_k(\overline{Q})$ is the k-th power sum (cf. [T2]). The second relation (6) comes from the equality

$$\widehat{c}(\overline{Q})\cdot\widehat{c}(\overline{S})=1+\widetilde{c}(\overline{\mathcal{E}}).$$

Here $\widetilde{c}(\overline{\mathcal{E}})$ is (the image in $CH(\overline{G})$ of) the *Bott-Chern form* of the exact sequence (3) for the total Chern class (cf. [BC] [GS2] [T2]). The Bott-Chern form $\widetilde{\phi}(\overline{\mathcal{E}})$ was computed in [T2] for any characteristic class ϕ:

$$\widetilde{\phi}(\overline{\mathcal{E}})=\sum_k\langle\phi,p_k\rangle\,\mathcal{H}_{k-1}p_{k-1}(\overline{Q})$$

where $\langle\ ,\ \rangle$ is the inner product of §2. For any symmetric function ϕ, homogeneous of degree r, this translates to the relation

$$\phi(\widehat{x},\widehat{y})=\langle\phi,p_r\rangle\,\mathcal{H}_{r-1}p_{r-1}(x) \tag{7}$$

in $CH(\overline{G})$. We will need to apply this result when $\phi=s_{\lambda/\mu}$ is a skew Schur function. In this case we have

Proposition 2 *$s_{\lambda/\mu}(\widehat{x},\widehat{y})=0$ unless λ/μ is a rim r-hook, in which case*

$$s_{\lambda/\mu}(\widehat{x},\widehat{y})=(-1)^{ht(\lambda/\mu)}\mathcal{H}_{r-1}p_{r-1}(x).$$

Proof. The argument is similar to the one used in [T2], Corollary 3. Assume $|\lambda/\mu|=r$. We start with the Frobenius formula

$$s_{\lambda/\mu}=\frac{1}{r!}\sum_{\sigma\in S_r}\chi_{\lambda/\mu}(\sigma)p_{(\sigma)}$$

where $\chi_{\lambda/\mu}$ is a generalized character and (σ) denotes the partition of r determined by the cycle structure of σ (cf. [M2], §I.7). Using (7) gives

$$s_{\lambda/\mu}(\widehat{x},\widehat{y}) = \chi_{\lambda/\mu}((12\ldots r))\mathcal{H}_{r-1}p_{r-1}(x).$$

The Murnaghan-Nakayama rule (cf. [JK] 2.4.7) is now used to compute the required value of $\chi_{\lambda/\mu}$:

$$\chi_{\lambda/\mu}((12\ldots r)) = \begin{cases} (-1)^{ht(\lambda/\mu)}, & \text{if } \lambda/\mu \text{ is a rim hook} \\ 0, & \text{otherwise.} \end{cases}$$

□

4 Arithmetic Schubert Calculus

4.1 Schubert calculus in $CH(\overline{G})$

Let us briefly review the classical Schubert calculus, which describes the multiplicative structure of $CH(G)$ for the Grassmannian $G = G(m,n)$. The abelian group $CH(G)$ is freely generated by the classes $s_\lambda(\mathrm{x}) = s_\lambda(Q)$, one for each λ contained in the $n \times m$ rectangle (m^n). (See Figure 2).

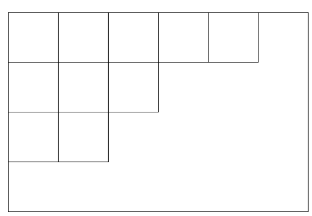

Figure 2: $m = 6$, $n = 4$ and $\lambda = (5,3,2)$

Under our notational conventions $s_{(1)^i}(\mathrm{x}) = e_i(\mathrm{x})$ is the i-th Chern class $c_i(Q)$ and is represented by the *special Schubert variety* X_i. More generally, $s_\lambda(\mathrm{x})$ is the class of the *Schubert variety* X_λ that parametrizes, over any base field k, subspaces $V \in G(k)$ such that $\dim(V \cap k^{n+i-\lambda'_i}) \geqslant i$, for $1 \leqslant i \leqslant m$.

It is important to interpret the isomorphism (4) in the following way: $CH(G)$ is isomorphic to the ring of Schur polynomials $s_\lambda(\mathrm{x})$ in the x variables modulo the ideal generated by the non-constant Schur polynomials $s_\lambda(\mathrm{x},\mathrm{y})$ in both sets of variables x, y. When we pass modulo this ideal the only Schur polynomials that survive are the $s_\lambda(\mathrm{x})$ for $\lambda \subset (m^n)$. This follows at once from the ideal property of Proposition 1.

For any two Schur functions there is a product formula

$$s_\lambda s_\mu = \sum_\nu N_{\lambda\mu}^\nu s_\nu \tag{8}$$

where the nonnegative integers $N_{\lambda\mu}^\nu$ are the *Littlewood-Richardson* coefficients. The *Pieri rule* for multiplying a Schur function by s_k, $k > 0$ is a special case of (8):

$$s_\lambda s_k = \sum_\mu s_\mu, \tag{9}$$

the sum over all μ obtained from λ by adding k boxes, with no two in the same column.

Recall the exact sequence (1) of §3. There are many splitting maps

$$\epsilon : CH(G) \to CH(\overline{G})$$

for this sequence; our choice is motivated by the generalization to flag varieties in [T1]. Define ϵ on the basis of Schur functions by $\epsilon(s_\lambda(\mathrm{x})) = s_\lambda(\widehat{x})$. This induces an isomorphism of abelian groups

$$CH(\overline{G}) \cong CH(G) \oplus \mathcal{H}(G_{\mathbb{R}}).$$

In other words, every element $z \in CH(\overline{G})$ has a unique expression

$$z = \sum_{\lambda \subset (m^n)} c_\lambda s_\lambda(\widehat{x}) + \sum_{\lambda \subset (m^n)} \gamma_\lambda s_\lambda(x),$$

where $c_\lambda \in \mathbb{Z}$ and $\gamma_\lambda \in \mathbb{R}$.

Since the alphabets $\widehat{x}$ and x have n variables, $e_k(\widehat{x})$ and $e_k(x)$ both vanish when $k > n$. Therefore the identity

$$s_\lambda = \det(e_{\lambda'_i - i + j})$$

(cf. [M2] §I (3.5)) implies that $s_\lambda(\widehat{x})$ and $s_\lambda(x)$ are zero whenever $l(\lambda) > n$. Note that if $\lambda_1 > m$ (so λ extends to the right of the rectangle (m^n)) then $s_\lambda(x) = 0$, but $s_\lambda(\widehat{x})$ need not vanish. In fact, $s_\lambda(\widehat{x})$ is the class of a differential form in $\mathcal{H}(G_{\mathbb{R}})$ which we will describe explicitly (see Proposition 3).

Based on the multiplication rule in §3 we see that for λ, μ in (m^n),

$$s_\lambda(\widehat{x}) \cdot s_\mu(x) = s_\lambda(x) s_\mu(x) = \sum_{\nu \subset (m^n)} N^\nu_{\lambda\mu} s_\nu(x)$$

and

$$s_\lambda(x) \cdot s_\mu(x) = 0.$$

It follows that the multiplication in $CH(\overline{G})$ will be completely characterized once the formula for multiplying two arithmetic Schubert classes $s_\lambda(\widehat{x}) \cdot s_\mu(\widehat{x})$ is known. From the relations in this ring we deduce that

$$s_\lambda(\widehat{x}) \cdot s_\mu(\widehat{x}) = \sum_{\substack{\nu \subset (m^n) \\ |\nu| = |\lambda| + |\mu|}} N^\nu_{\lambda\mu} s_\nu(\widehat{x}) + \sum_{\substack{\nu \subset (m^n) \\ |\nu| = |\lambda| + |\mu| - 1}} \widetilde{N}^\nu_{\lambda\mu}(m) s_\nu(x) \qquad (10)$$

Here the numbers $N^\nu_{\lambda\mu}$ are the classical Littlewood-Richardson numbers and $\widetilde{N}^\nu_{\lambda\mu}(m)$ are by definition the *arithmetic Littlewood-Richardson numbers*. The latter (a priori real) numbers were first defined by Maillot [Ma] in a different way. Although our notation and definition differs from the one in [Ma], these numbers are essentially the same (see §4.2).

Suppose λ and μ are two Young diagrams with $|\mu| = |\lambda| - 1$ and $r > 1$ is an integer. We define an r*-hook operation* from λ to μ to be the process of removing a rim r-hook from λ to get a diagram λ^-, followed by adding a rim $(r-1)$-hook to λ^- to obtain μ. We will show that there is at most one r-hook operation from λ to μ for any given r. The *sign* $\epsilon_{\lambda\mu}(r)$ of the operation is $+1$ (resp. -1) if the heights of the two rim hooks involved have the same (resp. opposite) parity mod 2. If there is no r-hook operation from λ to μ then we set $\epsilon_{\lambda\mu}(r) = 0$. Figure 3 illustrates a 6-hook operation from $\lambda = (6, 4, 3)$ to $\mu = (3^4)$ of positive sign.

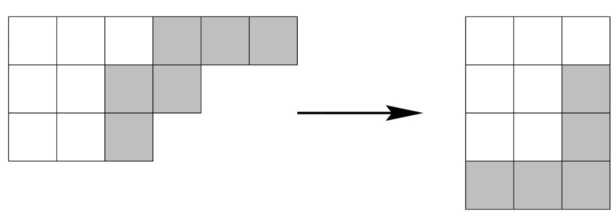

Figure 3: A hook operation from λ to μ

Following James and Kerber [JK], an r-hook operation can be conveniently visualized using sets of β-numbers, or β-sequences. For any partition

λ of length at most n, the *β-sequence* $\beta(\lambda)$ is defined as the n-tuple

$$\beta(\lambda) = (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n + n - n).$$

It is clear that the correspondence $\lambda \leftrightarrow \beta(\lambda)$ is 1-1, and that β-sequences consist of distinct integers. If $n = l(\lambda)$ then $\beta(\lambda)$ is the sequence $(h_{11}, h_{21}, \ldots, h_{n1})$ of first column hook lengths of λ. If $\beta = \beta(\lambda)$ then removing a rim r-hook from λ corresponds to changing a suitable β_i to $\beta_i - r$; reordering the resulting set of numbers produces a β-sequence for the new diagram (cf. [JK], Lemma 2.7.13).

We picture each β-sequence as a collection of n checkers on the squares of a semi-infinite horizontal strip, the checker positions corresponding to the numbers β_i (ordered as on the real line).

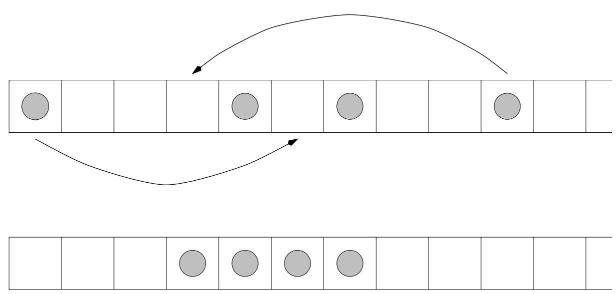

Figure 4: The same hook operation from $\beta(\lambda)$ to $\beta(\mu)$

In this picture an r-hook operation from λ to μ corresponds to moving a checker from $\beta(\lambda)$ r squares to the left, then moving a checker $r-1$ squares to the right to reach $\beta(\mu)$. Each move must be to an empty square. Note that the sign of the hook operation is determined by the total number of checkers 'jumped over'. Figure 4 shows the previous hook operation from $\beta(\lambda) = \{0, 4, 6, 9\}$ to $\beta(\mu) = \{3, 4, 5, 6\}$; in this example we have taken $n = 4$. From this description it is easy to see that for fixed λ, μ and r, there can be at most one such operation.

Define the rational number ξ_λ^μ by

$$\xi_\lambda^\mu = \sum_r \epsilon_{\lambda\mu}(r) \mathcal{H}_{r-1}.$$

We can now state our main result:

Theorem 1 *The arithmetic Littlewood-Richardson number $\widetilde{N}_{\lambda\mu}^{\nu}(m)$ is given by*

$$\widetilde{N}_{\lambda\mu}^{\nu}(m) = \sum_{\rho\,:\,\rho_1>m} \xi_{\rho}^{\nu} N_{\lambda\mu}^{\rho}. \tag{11}$$

Remark. Only partitions ρ such that there is a hook operation from ρ to ν contribute to the sum (11). The theorem implies that the arithmetic Littlewood-Richardson number $\widetilde{N}_{\lambda\mu}^{\nu}(m)$ is independent of n (note that it is only defined when $l(\nu) \leqslant n$). Indeed, it is easy to see that if $\rho_1 > m$ and $l(\rho) > l(\nu)$ then there is no hook operation from ρ to ν. Let $\mathcal{A}(m,n)$ denote the set of partitions ρ of length $\leqslant n$ such that the difference $\gamma_\rho = \rho \smallsetminus (m^n)$ is a hook of height $ht(\gamma_\rho) \leqslant n-2$ and of length at most $m+n-1-ht(\gamma_\rho)$. It is easily verified that if $\rho_1 > m$ and $\rho \notin \mathcal{A}(m,n)$ then a hook operation on ρ cannot lead to a partition $\nu \subset (m^n)$. Thus only $\rho \in \mathcal{A}(m,n)$ can contribute to the sum in Theorem 1.

From the theorem one may deduce the following *arithmetic Pieri rule*: For $1 \leqslant k \leqslant n$ let $B(\lambda,k)$ be the set of partitions obtained from λ by adding k boxes, with no two in the same row. Then for $\lambda \subset (m^n)$ we have

$$s_\lambda(\widehat{x}) \cdot s_{(1^k)}(\widehat{x}) = \sum_{\mu} s_\mu(\widehat{x}) + \sum_{\nu} \left(\sum_{\rho} \xi_{\rho}^{\nu} \right) s_\nu(x).$$

Here the first (classical) sum is over $\mu \in B(\lambda,k)$ with $\mu_1 \leqslant m$ and the second sum is over ν and ρ with $\rho \in B(\lambda,k)$ and $\rho_1 > m$. Note that the second sum vanishes unless $\lambda_1 = m$.

Example. This example shows that arithmetic Littlewood-Richardson numbers need not be positive, as well as exhibiting their dependence on m. Consider $\lambda = (2,2)$, $\mu = (2,1)$ and $\nu = (2,2,1,1)$. There is a hook operation from each of the partitions $\rho_1 = (4,3)$, $\rho_2 = (4,2,1)$, $\rho_3 = (3,3,1)$ and $\rho_4 = (3,2,1,1)$ to ν (in fact exactly two from each), and these are the only such partitions that appear in the product $s_\lambda \cdot s_\mu$. The classical Littlewood-Richardson coefficients are

$$N_{\lambda\mu}^{\rho_1} = N_{\lambda\mu}^{\rho_2} = N_{\lambda\mu}^{\rho_3} = N_{\lambda\mu}^{\rho_4} = 1.$$

Theorem 1 gives

$$\widetilde{N}_{\lambda\mu}^{\nu}(3) = \xi_{\rho_1}^{\nu} N_{\lambda\mu}^{\rho_1} + \xi_{\rho_2}^{\nu} N_{\lambda\mu}^{\rho_2} = (\mathcal{H}_2 - \mathcal{H}_4) + (\mathcal{H}_1 - \mathcal{H}_5) = -\frac{28}{15},$$

$$\widetilde{N}^{\nu}_{\lambda\mu}(2) = (\mathcal{H}_2 - \mathcal{H}_4) + (\mathcal{H}_1 - \mathcal{H}_5) + (\mathcal{H}_4 - \mathcal{H}_1) + (\mathcal{H}_2 + \mathcal{H}_5) = 3.$$

In the same example $\widetilde{N}^{\nu}_{\lambda\mu}(m) = 0$ for each $m \geqslant 4$.

Proof of Theorem 1. We begin with the formal identity

$$s_\lambda(\widehat{x}) = \sum_{\mu \subset (m^n)} (-1)^{|\mu|} s_{\lambda/\mu}(\widehat{x}, \widehat{y}) \cdot s_{\mu'}(\widehat{y}) \tag{12}$$

from Proposition 1. Assume $\lambda_1 > m$, so that all polynomials $s_{\lambda/\mu}$ in this sum either vanish or have positive degree. Furthermore we know that

$$s_{\mu'}(y) = (-1)^{|\mu|} s_\mu(x)$$

in $\mathcal{H}(G_{\mathbb{R}})$ (cf. [Fu1] Lemma 14.5.1). Using this and Proposition 2, (12) becomes

$$s_\lambda(\widehat{x}) = \sum_{\mu} (-1)^{ht(\lambda/\mu)} \mathcal{H}_{r(\mu)-1} p_{r(\mu)-1}(x) s_\mu(x), \tag{13}$$

the sum over all $\mu \subset (m^n)$ such that λ/μ is an rim $r(\mu)$-hook. There is a general rule for multiplying a partition by a power sum p_r (cf. [M2] §I.3, Ex. 11); this states that

$$p_r s_\mu = \sum_{\nu} (-1)^{ht(\nu/\mu)} s_\nu, \tag{14}$$

the sum over all $\nu \supset \mu$ such that ν/μ is a rim r-hook. Now combine (13) and (14) to get

Proposition 3 *For partitions λ with $\lambda_1 > m$ we have*

$$s_\lambda(\widehat{x}) = \sum_{\nu} \xi^{\nu}_{\lambda} s_\nu(x), \tag{15}$$

the sum over all $\nu \subset (m^n)$ that can be obtained from λ by a hook operation.

Using Proposition 3 and the previous Remark we see that if $\lambda_1 > m$ and $\lambda \notin \mathcal{A}(m.n)$, then $s_\lambda(\widehat{x}) = 0$. The proof of the theorem is completed by writing the identity

$$s_\lambda(\widehat{x}) \cdot s_\mu(\widehat{x}) = \sum_{\substack{\nu \subset (m^n) \\ |\nu| = |\lambda| + |\mu|}} N^{\nu}_{\lambda\mu} s_\nu(\widehat{x}) + \sum_{\substack{\rho \,:\, \rho_1 > m \\ |\rho| = |\lambda| + |\mu|}} N^{\rho}_{\lambda\mu} s_\rho(\widehat{x}),$$

using (15) to replace the classes in the second sum, collecting terms, and comparing with (10). □

4.2 Duality

We discuss here the effect of the canonical duality isomorphism of $G(m,n)$ with $G(n,m)$ on the formulas of the previous section. This isomorphism takes the Schubert variety X_λ to $X_{\lambda'}$, for $\lambda \subset (m^n)$. It induces an isomorphism

$$CH(\overline{G}(m,n)) \cong CH(\overline{G}(n,m))$$

of Arakelov Chow rings. The fact that this map is an isomorphism is not obvious from the presentation given in §3. An algebraic proof of this is given in §6 in a more general setting. Our purpose in this section is to relate our work to that of Maillot [Ma].

Define the symmetric function $s = \sum s_i$. The relation (7) in $CH(\overline{G}(m,n))$ for $\phi = s$ gives

$$s(\widehat{x}) \cdot s(\widehat{y}) = 1 + \sum \mathcal{H}_k p_k(x).$$

Since $p_k(x) + p_k(y) = p_k(x,y) = 0$, we may rewrite this equation as

$$s(\widehat{x}) \cdot s(\widehat{y}) \cdot (1 + \sum \mathcal{H}_k p_k(y)) = 1$$

or

$$s(\widehat{x}) \cdot s(\widehat{y}\,|\,y) = 1, \tag{16}$$

where

$$s_k(\widehat{y}\,|\,y) := s_k(\widehat{y}) + \sum_{i+j=k-1} \mathcal{H}_i p_i(y) s_j(y).$$

(compare with §6 and [Ma], §5.2). The duality isomorphism, regarded as an involution on $CH(\overline{G}(m,n))$, sends $s_\lambda(x)$ to $s_{\lambda'}(y)$. It follows from equation (16) that $s_\lambda(\widehat{x})$ is sent to $s_{\lambda'}(\widehat{y}\,|\,y)$. We conclude that the image of the multiplication rule (10) under the duality map is

$$s_{\lambda'}(\widehat{y}\,|\,y) \cdot s_{\mu'}(\widehat{y}\,|\,y) = \sum_{\substack{\nu \subset (m^n) \\ |\nu| = |\lambda| + |\mu|}} N_{\lambda\mu}^{\nu} s_{\nu'}(\widehat{y}\,|\,y) + \sum_{\substack{\nu \subset (m^n) \\ |\nu| = |\lambda| + |\mu| - 1}} \widetilde{N}_{\lambda\mu}^{\nu}(m) s_{\nu'}(y). \tag{17}$$

A comparison of (17) with Theorem 5.2.1 of [Ma] shows that the arithmetic Littlewood-Richardson coefficients defined by Maillot coincide with our numbers $\widetilde{N}_{\lambda\mu}^{\nu}(m)$ under the duality involution. To obtain the formulas in [Ma] one maps s_p(x) (resp. s_q(y)) to the p-th Chern class of the universal subbundle (resp. the q-th Chern class of the universal quotient bundle) throughout.

5 Height Calculations

5.1 The height of $G(m,n)$

In this section we apply the results of §4 to compute the Faltings height of $G = G(m,n)$ under the Plücker embedding. This number was first calculated by Maillot [Ma]; our formula is an improvement of his. We begin by recalling some combinatorics:

A *standard tableau* on the Young diagram λ is a numbering of the boxes of λ with the integers $1, 2, \ldots, |\lambda|$ such that the entries are strictly increasing along each row and column. The number of standard tableaux on λ is denoted f^λ and is given by the elegant hook length formula

$$f^\lambda = \frac{|\lambda|!}{\prod_{x\in\lambda} h_x} = |\lambda|! \frac{\prod_{i<j}(\beta_i - \beta_j)}{\prod \beta_i!} \tag{18}$$

where $\{\beta_i\}$ is the β-sequence of λ (cf. [M2] Examples I.1.1 and I.5.2). Note that iterating the Pieri rule (9) gives

$$s_1^r = \sum_{|\lambda|=r} f^\lambda s_\lambda. \tag{19}$$

The Grassmannian G has a natural Plücker embedding in projective space given by the very ample line bundle $\det Q$. In geometry the the *degree* of $G(k)$ (for any field k) under this embedding is given by

$$\deg(G(k)) = f^{(m^n)}. \tag{20}$$

This follows from equation (19). The height of G is an arithmetic analogue of this number; our formulas will be 'arithmetic perturbations' of (20).

Let $\overline{\mathcal{O}}(1)$ denote the canonical line bundle on projective space equipped with the invariant metric (so that $c_1(\overline{\mathcal{O}}(1))$ is the Fubini-Study form). The *height* of G under the Plücker embedding, as defined by Faltings [F], is the number

$$ht_{\overline{\mathcal{O}}(1)}(G) = \widehat{\deg}(\widehat{c}_1(\overline{\mathcal{O}}(1))^d \,|\, G) = \widehat{\deg}(s_1^d(\widehat{x})). \tag{21}$$

Here the arithmetic degree map $\widehat{\deg}$ is defined as in [BoGS] and $d = mn+1$ is the absolute dimension of G. In $CH(\overline{G})$ the arithmetic intersection

$$s_1^d(\widehat{x}) = r_d s_{(m^n)}(x) = r_d s_{(m^n)}(\overline{Q})$$

for a rational number r_d; the height (21) is then given by

$$ht_{\overline{\mathcal{O}}(1)}(G) = \frac{1}{2}\int_{G(\mathbb{C})} r_d s_{(m^n)}(\overline{Q}) = \frac{r_d}{2} \tag{22}$$

as $s_{(m^n)}(\overline{Q})$ is dual to the class of a point in $G(\mathbb{C})$.

To compute this number we will use the machinery developed in §4. First define the following *fundamental set of diagrams*:

$$\mathcal{D}(m,n) = \{\lambda \in \mathcal{A}(m,n) \ : \ |\lambda| = d\} = \{\lambda_0\} \cup \{[a,b,\gamma(i,j)]\}_{(a,b,i,j)\in I}$$

Here

$$\lambda_0 = (m^n) + (1)$$

and

$$[a,b,\gamma(i,j)] = ((m-1)^{n-1}, a) + (1^b) + \gamma(i,j)$$

where $\gamma(i,j) = (i, 1^j)$ is a hook, and the indexing set $I = I(m,n)$ of 4-tuples (a,b,i,j) is defined by the conditions

$$0 \leqslant a < m \ , \ 0 \leqslant j < b < n \ , \quad i = m+n-a-b-j > 0.$$

(see Figure 5).

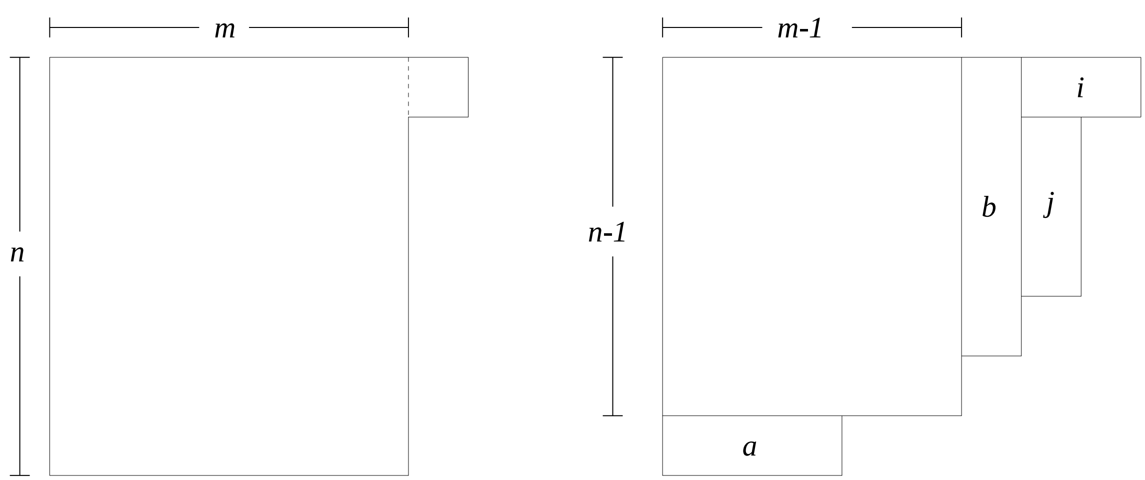

Figure 5: λ_0 and $[a,b,\gamma(i,j)]$

For $s > r > 0$ natural numbers let

$$\mathcal{H}_r^s = \mathcal{H}_{s-1} - \mathcal{H}_{r-1} = \frac{1}{r} + \cdots + \frac{1}{s-1}$$

denote the difference of harmonic numbers. Then we have

Theorem 2 *The height of the Grassmannian $G(m,n)$ in its Plücker embedding is*

$$ht_{\overline{\mathcal{O}}(1)}(G(m,n)) = \frac{1}{2}\left(\sum_{k=n}^{m+n-1}\mathcal{H}_k\right) f^{\lambda_0} + \frac{1}{2}\sum_{(a,b,i,j)\in I}(-1)^{n+b+j}\mathcal{H}_{i+j}^{i+b} f^{[a,b,\gamma(i,j)]}$$

and is a number in $\sum_{k=1}^{m+n-1}(\frac{1}{2k}\mathbb{Z})$.

Proof. We begin by using identity (19) to get

$$s_1^d(\widehat{x}) = \sum_{|\rho|=d} f^\rho s_\rho(\widehat{x}). \tag{23}$$

Now Proposition 3 is applied to evaluate the classes $s_\rho(\widehat{x})$ with $|\rho| = d$. Note that $s_\rho(\widehat{x})$ will vanish unless there is a hook operation from ρ to (m^n). In the latter case ρ must have a box in the $(m-1, n-1)$ position; this is equivalent to $(m-1)^{n-1} \subset \rho$. It follows that $s_\rho(\widehat{x}) \neq 0$ if and only if $\rho \in D(m,n)$, so (23) may be written

$$s_1^d(\widehat{x}) = f^{\lambda_0} s_{\lambda_0}(\widehat{x}) + \sum_{(a,b,i,j)\in I} f^{[a,b,\gamma(i,j)]} s_{[a,b,\gamma(i,j)]}(\widehat{x}). \tag{24}$$

A hook operation from any $\rho \in D(m,n)$ to (m^n) must begin by removing one of the $m+1$ rim hooks $R_{11}, \ldots, R_{1,m+1}$ corresponding to the first $m+1$ boxes in the first row of ρ. From this we see that there are m such operations starting from λ_0 (all of sign $+1$), while only two (with opposite signs) from each diagram $[a,b,\gamma(i,j)]$. Figure 6 illustrates the rim hooks removed in the four hook operations from $\lambda_0 = (5, 4^5)$ and the two hook operations from $[1, 5, \gamma(3,1)] = (7,5,4,4,4,1)$ to $(m^n) = (4^6)$.

It follows that

$$\xi_{\lambda_0}^{(m^n)} = \sum_{k=n}^{m+n-1}\mathcal{H}_k \tag{25}$$

and

$$\xi_{[a,b,\gamma(i,j)]}^{(m^n)} = (-1)^{n+b+j}(\mathcal{H}_{i+b-1} - \mathcal{H}_{i+j-1}) = (-1)^{n+b+j}\mathcal{H}_{i+j}^{i+b}. \tag{26}$$

We now use (25) and (26) in Proposition 3 to express (24) as a scalar multiple of $s_{(m^n)}(x)$; applying (22) then completes the proof. □

Notice that it is not clear from Theorem 2 that the formula for $ht(G(m,n))$ is symmetric in m and n, even for projective space! This is because the partitions in $\mathcal{D}(m,n)$ are not conjugate to those in $\mathcal{D}(n,m)$. We discuss this further in §6.

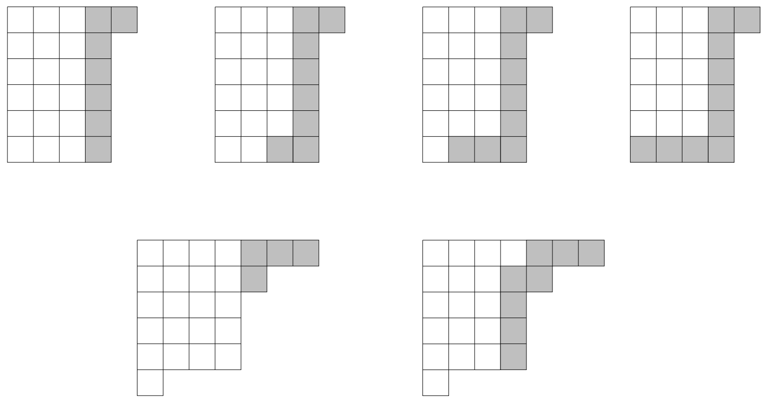

Figure 6: Hook operations leading to (4^6)

5.2 An example: $G(2, n)$

The formulas of the previous section can be simplified further in the special case when m or n equals 2. Either specialization will give the same height, by duality, however it will be combinatorially simpler to work with the case $n = 2$.

To calculate the height of $G(m, 2)$ we must compute $s_1(\widehat{x})^{2m+1}$. In this case we have

$$\mathcal{D}(m, 2) = \{\lambda_k = (m + k, m + 1 - k) \mid 1 \leqslant k \leqslant m + 1\}.$$

It follows from formula (18) that

$$f^{\lambda_k} = \frac{2k}{m + k + 1} \binom{2m + 1}{m + k}$$

so Theorem 2 gives

$$ht_{\overline{\mathcal{O}}(1)}(G(m, 2)) = \sum_{k=2}^{m+2} \left(\frac{\mathcal{H}_k}{m + 2} \binom{2m + 1}{m + 1} - \frac{1}{m + k + 1} \binom{2m + 1}{m + k} \right). \quad (27)$$

One can evaluate the sum (27) by using the three identities

$$\sum_{k=1}^{r} \mathcal{H}_k = (r + 1)\mathcal{H}_r - r$$

$$\sum_{k=0}^{r} \frac{1}{k+1}\binom{r}{k} = \frac{2^{r+1}-1}{r+1}$$

$$\frac{1}{m+k+1}\binom{2m+1}{m+k} = \frac{1}{m-k+1}\binom{2m+1}{m-k}.$$

After simplification we arrive at

Corollary 1

$$ht_{\overline{\mathcal{O}}(1)}(G(2,n)) = ht_{\overline{\mathcal{O}}(1)}(G(n,2)) = \left(\mathcal{H}_{n+2} - \frac{2n+1}{2n+2}\right)\binom{2n+1}{n} - \frac{4^n}{n+1}.$$

6 The combinatorics of duality

We begin this section by observing that the dependence of the formulas of §3–§5 on harmonic numbers is linear. This occurs because these numbers live in a square zero ideal of $CH(\overline{G})$. The algebraic and combinatorial theory developed thus far is equally valid if we replace them by a different sequence of real numbers. This allows us to break down the multiplicative structure into its essential combinatorial units.

In fact, we intend to work with a more general class of rings. Let $\{\widehat{x},\widehat{y}\}$ and $\{x,y\}$ be alphabets of cardinalities $n+m$ each and $\mathcal{A}=\mathbb{Z}[\widehat{x},\widehat{y}]^H$, $\mathcal{B}=\mathbb{R}[x,y]^H$ and the multiplication $\cdot$ on $\mathcal{A}\oplus\mathcal{B}$ be defined as in §3. Define $R[\widehat{x},\widehat{y},x,y]$ to be the ring $(\mathcal{A}\oplus\mathcal{B},\cdot)$ modulo the two relations

$$s_k(x,y) = 0 \tag{28}$$

and

$$s_k(\widehat{x},\widehat{y}) = \alpha_{k-l}p_{k-l}(x) \tag{29}$$

for all $k \geqslant 1$. Here $\{\alpha_j\}$ is a sequence of real numbers and l is any integer. When $l=1$ and $\alpha_j = \mathcal{H}_j$ we obtain the Arakelov Chow ring of §3.

There is a 'Schubert calculus' in $R[\widehat{x},\widehat{y},x,y]$ formally identical to the one for $CH(\overline{G})$. In fact all our proofs are algebraic and combinatorial; for instance the key relation

$$\phi(\widehat{x},\widehat{y}) = \langle \phi, p_k\rangle\, \alpha_{k-l}p_{k-l}(x) \tag{30}$$

holds, for any symmetric function ϕ, homogeneous of degree k. This follows from the dual of Newton's identity

$$ks_k = p_1s_{k-1} + \cdots + p_{k-1}s_1 + p_k$$

as in [T2] §5.

For λ and μ two Young diagrams with $|\mu| = |\lambda| - l$ and $r > l$ define an (r,l)-hook operation from λ to μ to be the process of removing a rim r-hook from λ to reach a diagram λ^-, followed by adding a rim $(r-l)$-hook to λ^- to obtain μ. The sign $\epsilon_{\lambda\mu}(r,l)$ of the operation is defined as in §4.1; also let

$$\xi_\lambda^\mu := \sum_r \epsilon_{\lambda\mu}(r,l)\alpha_{r-l}. \tag{31}$$

There are analogues $N_{\lambda\mu}^\nu(l,m,n)$ of the arithmetic Littlewood-Richardson coefficients, defined as before; notice that for $l > 1$ these numbers may depend on both m and n. Replacing 'r-hook operation' by '(r,l)-hook operation' and $\mathcal{H}_k$ by α_k in the results of §4 and §5 gives valid formulas in $R[\widehat{x},\widehat{y},x,y]$, although the analogues of the height calculations in §5 lack arithmetic significance.

Now consider a dual ring $R'[\widehat{u},\widehat{v},u,v]$ where the alphabets $\widehat{u}$, u (resp. $\widehat{v}$, v) have m (resp. n) variables each. The multiplication in R' is defined in the same manner as for R, except that the relation (29) is replaced by

$$s_k(\widehat{u},\widehat{v}) = (-1)^{l+1}\alpha_{k-l}p_{k-l}(u). \tag{32}$$

Following §4.2, if we let

$$s_k(\widehat{u}\,|\,u) := s_k(\widehat{u}) + (-1)^l \sum_{j<k} \alpha_{k-l-j}p_{k-l-j}(u)s_j(u) \tag{33}$$

then (32) summed over all k may be written

$$s(\widehat{u}\,|\,u)\cdot s(\widehat{v}) = 1,$$

which implies that

$$s_\lambda(\widehat{u}\,|\,u) = (-1)^{|\lambda|}s_{\lambda'}(\widehat{v}) \tag{34}$$

for each Young diagram λ. Our aim is to prove

Proposition 4 *The map $\varphi : R[\widehat{x},\widehat{y},x,y] \to R'[\widehat{u},\widehat{v},u,v]$ defined by*

$$\varphi(s_\lambda(x)) = s_{\lambda'}(u) \qquad \text{and} \qquad \varphi(s_\lambda(\widehat{x})) = s_{\lambda'}(\widehat{u}\,|\,u)$$

is a ring isomorphism.

Proof. We first verify that φ is well defined. Relation (28) is clearly preserved by φ, so we are left with checking the image of (29). By definition,

$$\varphi(s_k(\widehat{x})) = s_{(1^k)}(\widehat{u} \,|\, u) = (-1)^k s_k(\widehat{v}).$$

We claim that $\varphi(s_k(\widehat{y})) = (-1)^k s_k(\widehat{u})$. To see this, use Proposition 1 to get

$$\begin{aligned} s_k(\widehat{y}) &= \sum_j (-1)^j s_{k-j}(\widehat{x},\widehat{y}) s_{(1^j)}(\widehat{x}) \\ &= (-1)^k s_{(1^k)}(\widehat{x}) + \sum_{j<k} (-1)^j \alpha_{k-j-l} p_{k-j-l}(x) s_{(1^j)}(x). \end{aligned}$$

Thus we have

$$\begin{aligned} \varphi(s_k(\widehat{y})) &= (-1)^k s_k(\widehat{u} \,|\, u) + \sum_{j<k} (-1)^{k-l-1} \alpha_{k-j-l} p_{k-j-l}(u) s_j(u) \\ &= (-1)^k s_k(\widehat{u}). \end{aligned}$$

Now we can compute

$$\varphi(s_k(\widehat{x},\widehat{y})) = \sum_{p+q=k} \varphi(s_p(\widehat{x}))\varphi(s_q(\widehat{y})) = (-1)^k s_k(\widehat{u},\widehat{v}). \tag{35}$$

For the right hand side of (29), observe that the definition of φ gives

$$\varphi(p_{k-l}(x)) = (-1)^{k-l-1} p_{k-l}(u). \tag{36}$$

Use (35) and (36) in relation (29) to complete the argument that φ is well defined. Since φ maps the generators $s_\lambda(x)$ to $(-1)^{|\lambda|} s_\lambda(v)$ and $s_\lambda(\widehat{x})$ to $(-1)^{|\lambda|} s_\lambda(\widehat{v})$, it is clear that φ is a ring isomorphism. □

We have shown that φ gives an involution on R exactly when the integer l is odd; if l is even then the dual ring R' differs by a sign. We can define more general rings than these by writing other symmetric functions on the right hand side of (29), but the dual picture will differ even more.

The duality isomorphism leads to interesting combinatorial identities. For these, we may set $\alpha_j = \delta_{l+j,r}$ for some fixed r (here δ is the Kronecker delta). This amounts to restricting to (r, l)-hook operations for fixed r and l, with each operation contributing a factor equal to its sign. In this case we have

$$\xi_\lambda^\mu = \epsilon_{\lambda\mu}(r, l)$$

where $\epsilon_{\lambda\mu}(r,l)=0$ if there is no (r,l)-hook operation from λ to μ.

Let $\mathcal{A}_l(m,n)$ denote the set of partitions ν of length at most n such that the difference $\nu \smallsetminus (m^n)$ is a hook $(i,1^j)$ of height $j \leqslant n-2$ and such that $i+2j+2 \leqslant l+m+n$. If $l(\nu) \leqslant n$, $\nu_1 > m$ and $\nu \notin \mathcal{A}_l(m,n)$ then there is no (r,l)-hook operation from ν to any diagram contained in (m^n).

Note that for partitions λ with $(1,1)$ hook length $h_{11}(\lambda) \leqslant l$, we have $\varphi(s_\lambda(\widehat{x})) = s_{\lambda'}(\widehat{u})$. Indeed, for such λ all terms $s_k(\widehat{u}\,|\,u)$ that occur in the determinant for $s_{\lambda'}(\widehat{u}\,|\,u)$ satisfy $s_k(\widehat{u}\,|\,u) = s_k(\widehat{u})$. Although one can take $\{\mu_i\}$ to be any sequence of such partitions in discussion that follows, we restrict to single row partitions in order to connect the results with the classical Kostka numbers.

Recall that a *tableau of content* $\mu=(\mu_1,\ldots,\mu_k)$ on the Young diagram λ is a numbering of the boxes of λ with μ_1 1's, μ_2 2's, … , μ_k k's, which are weakly increasing across rows and strictly increasing down columns. The number of tableaux on λ with content μ is given by the *Kostka number* $K_{\lambda\mu}$. Equivalently, we have the following equations:

$$\begin{aligned} s_{\mu_1}\cdots s_{\mu_k} &= \sum_\lambda K_{\lambda\mu}s_\lambda \\ s_{(1^{\mu_1})}\cdots s_{(1^{\mu_k})} &= \sum_\lambda K_{\lambda'\mu}s_\lambda \end{aligned}$$

(cf. [Fu3] §2.2). We can now state

Proposition 5 *Suppose that $\mu_1 \leqslant l$ and $|\mu| > mn$. Then for any given partition $\nu \subset (m^n)$,*

$$\sum_{\lambda\in\mathcal{A}_l(m,n)} \epsilon_{\lambda\nu}(r,l)K_{\lambda\mu} = (-1)^{l+1} \sum_{\lambda\in\mathcal{A}_l(n,m)} \epsilon_{\lambda\nu'}(r,l)K_{\lambda'\mu}.$$

Proof. The relation

$$s_{\mu_1}(\widehat{x})\cdots s_{\mu_k}(\widehat{x}) = \sum_\lambda K_{\lambda\mu}s_\lambda(\widehat{x}) = \sum_{\nu\subset(m^n)}\left(\sum_\lambda \epsilon_{\lambda\nu}(r,l)K_{\lambda\mu}\right)s_\nu(x)$$

in $R[\widehat{x},\widehat{y},x,y]$ is sent to

$$s_{(1^{\mu_1})}(\widehat{u})\cdots s_{(1^{\mu_k})}(\widehat{u}) = \sum_\lambda K_{\lambda'\mu}s_\lambda(\widehat{u}) = (-1)^{l+1}\sum_{\rho\subset(n^m)}\left(\sum_\lambda \epsilon_{\lambda\rho}(r,l)K_{\lambda'\mu}\right)s_\rho(u)$$

under the isomorphism φ. Since $\varphi(s_\nu(x)) = s_{\nu'}(u)$, the result follows. $\square$

If $\mu = (1^k)$ then $K_{\lambda\mu} = K_{\lambda'\mu} = f^\lambda$ for each partition λ of weight k. In this case Proposition 5 specializes to

Corollary 2 *Let $\nu \subset (m^n)$ be a partition such that $|\nu| + l > mn$. Then*

$$\sum_{\lambda \in \mathcal{A}_l(m,n)} \epsilon_{\lambda\nu}(r,l) f^\lambda = (-1)^{l+1} \sum_{\lambda \in \mathcal{A}_l(n,m)} \epsilon_{\lambda\nu'}(r,l) f^\lambda.$$

Both the Proposition and the Corollary give nontrivial relations among the classical Kostka numbers. We do not know a correspondence among Young tableaux that would explain either of these results.

Example. In the special case when $l = 1$ the only partition $\nu \subset (m^n)$ with $|\nu| + l > mn$ is (m^n) itself, so Corollary 2 states that

$$\sum_{\lambda \in \mathcal{A}(m,n)} \epsilon_{\lambda(m^n)}(r) f^\lambda = \sum_{\lambda \in \mathcal{A}(n,m)} \epsilon_{\lambda(n^m)}(r) f^\lambda$$

where $\epsilon_{\lambda\mu}(r) = \epsilon_{\lambda\mu}(r,1)$ is as in §4.1. For $m = 3$, $n = 2$ and $r = 4$ this relation becomes

$$f^{(4,3)} - f^{(6,1)} + f^{(7)} = f^{(3,2,2)} - f^{(4,2,1)} + f^{(5,1,1)} + f^{(4,3)} - f^{(6,1)} = 9.$$

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