A Filtering Framework for Time-Varying Graph Signals

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Abstract Time-varying graph signal processing generalizes scalar graph signals to multivariate time-series data with an underlying graph structure. Important applications include network neuroscience, social network analysis, and sensor processing. In this chapter, we present a framework for modeling the underlying graphs of these multivariate signals along with a filter design methodology based on invariance to the graph-shift operator. Importantly, these approaches apply to directed and undirected graphs. We present three classes of filters for time-varying graph signals, providing example application of each in design of ideal bandpass filters.

1 Introduction

The field of graph signal processing has seen rapid growth since its introduction just five years ago in the works of Sandryhaila and Moura [33, 35] and Shuman et al. [38]. Much of the graph signal processing work can be understood as migrating the theoretical underpinnings of classical signal processing of time-series data to that of scalar signals defined on the nodes of a graph. Not all of this work is within the

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scope of this chapter. For a comprehensive review of the developments, applications, and open problems in graph signal processing more generally, see Ortega et al. [29].

In the present chapter, we consider the modeling and filtering time-varying graph signals on possibly directed graphs. Time-varying graph signals can be understood as multivariate stochastic processes for which the multivariate components have a topology induced by a graph structure. Unlike conventional graph signal processing with scalar signals on graphs, time-varying graph signals also incorporate a temporal dimension. Here, we consider how to model the underlying directed (or undirected) graph along with the design of filters for such signals. Our filtering construction is motivated by covariance to the underlying graph shift operator. Target applications for such modeling and filtering are network neuroscience, social network analysis, and sensor array processing. Each of these applications has a physical topology induced by an underlying graph, while the signals are revealed in time-series. For example, the members of a social network comprise the nodes of a graph with edges connecting friends or followers, and the recorded signal could be network activity (e.g. sending messages or updating status) indexed by time.

One notable application of graph signal processing has been its use in deep learning architectures. Deep learning encompasses a diverse field of statistical models and learning algorithms which have achieved state-of-the-art results in image, video, speech, and language applications [22]. The seminal work by Mallat [26, 27] and Bruna and Mallat [3] argued that the power of deep learning to achieve invariances necessary for generalization could be understood as the result of composing conventional wavelet filter banks with non-linear functions. This led to the first work to generalize deep learning to graph signals by Bruna et al [4]. In this work, the convolution operators in deep learning architectures are replaced with spectrally-defined graph filters consistent with the graph Fourier transform of Shuman et al [38]. This work inspired a thread of follow-on research [14, 6, 18] which is reviewed in detail in Bronstein et al. [1].

At its foundation, the field of graph signal processing has had a fundamental divide. The theory proposed by Sandryhaila and Moura [33, 35] advocates an algebraic construction of graph signal processing from the weighted adjacency matrix. Importantly, this theory accommodates directed graphs. However, the theory proposed by Shuman et al [38] advocates a construction of graph signal processing from the definition of the graph Fourier transform by means of the graph Laplacian. Importantly, the graph Laplacian for an undirected graph is symmetric and positive semi-definite. This yields an orthonormal basis in the definition of the graph Fourier transform along with imparting a physical intuition for the eigenvectors. Underlying this debate is the lack of a canonical definition for a graph-shift and the consequences of choosing one over another [25]. Later work has advocated that the graph-shift be an isometry [11, 9], a stochastic matrix [43], or have a uniquely-defined orthonormal basis [36]. To add to the confusion, Deri and Moura point-out the inherent ambiguity in choosing eigenvectors from the invariant subspaces of matrices with semi-simple and degenerate eigenvalues [7].

Regarding extensions of graph signal processing to time-varying signals on graphs, there are two primary thrusts. One follows from a desire to track random processes on graphs. Auto-regressive moving average models for tracking time-varying signals on graphs are proposed in [24] and [16], and reconstruction techniques for sub-sampled time-varying graph signals are proposed in [42, 31, 15, 32]. The other collection of works relates more closely to the proposed approach of this chapter. These works consider multi-dimensional signals on graphs as factor graphs. Sandryhaila and Moura first proposed this approach in [34] for a general factor graph framework and discussed the application of time-varying signals on a graph. A similar concept was proposed for a generalized analysis of multi-dimensional graph signals recently in [21]. In the interim, Loukas and Foucard and Grassi et al. have proposed a joint Fourier transform and associated analysis of time-varying signals on graphs in [23] and [13] respectively.

Other works which address time-varying signals on graphs is that of Villafane-Delgado et al. which considers the tensor decomposition of a Laplacian tensor defined over all time [41]. In Yan et al. [43], a parameterized filter is defined which allows the filtering of continuous time processes on graphs. Smith et al. [40] propose methods by which to define correlation, coherence, and phase-lag index for time-varying signals.

The current chapter shares a common motivation with that of Bruna et al. [4] with regard to designing filters for a machine learning application. It is also similar in its state goals to the optimal filtering framework of Segarra et al. [37], but for the focus on time-varying graph signals. The definition of shift-invariance used in the current work first appeared in Sandryhaila and Moura [33] without the context of time. The result of Theorem 2 is very similar to notions of graph stationarity proposed in Girault [10] and Marques et al. [28], but stationarity is understood to be a property of the graph signal and not a property of the filter as in the current chapter. Romero et al. [32] propose an extended graph formulation very similar to that introduced in Section 2 of the current chapter. The extended graph formulation of Romero et al. is used to define regularization terms for recovering time-varying signals on graphs. However, these works are not primarily concerned with analysis and filtering as is the focus of this chapter. The works of Loukas and Foucard [23] and Grassi et al. [13] are seen to be the most closely related in scope and purpose to the current chapter. These works directly address the analysis and filtering of time-varying signals on graphs. However, these works consider only undirected graphs and use the factor graph approach which does not allow edges across multiple time scales as in the current chapter.

In Sandryhaila and Moura [34], the authors propose a procedure for filtering time-varying graph signals by modeling the underlying graph as one of the various graph products between the circulant shift operator and the weighted adjacency matrix of the graph nodes. This formulation allows the authors to apply both the tools of discrete signal processing and of graph signal processing disjointly on the problem. Similar models have been proposed in Loukas and Foucard [23], Grassi et al. [13], and Kurokawa et al. [21].

Implicit in the factor graph model is a strong Markov assumption about how the nodes of a graph interact in time. The factor graph models only the interaction among nodes at a fixed time-scale. This is not overly limiting, but it misses a more full modeling of the possible interactions among nodes in time. For instance, two nodes could interact via an unmodeled process, such as a node that is not included in the graph. This interaction may manifest over two, three, or more time-steps. Simultaneously, other nodes may interact at a single time-step. Additionally, the interaction between nodes may be cumulative or history-dependent. Neither of these cases fit into the factor graph model of time-series graph signals.

This chapter aims to address this gap by modeling stationary interactions between nodes in time. We propose a theoretical framework for linear filtering of time-varying graph signals on directed or undirected time-invariant graphs. This framework yields a family of filter design methods of decreasing design complexity (see Table 1). The proposed filter design methods will be motivated by a common example application of bandpass filtering.

Section 2 covers some preliminaries which establish the extended graph framework and the function spaces in which time-varying graph signals exist. Section 3 adapts linear and time-invariant systems theory to a graph geometry. Using this framework, filter design procedures are derived which yield quadratic complexity. Section 4.2 derives shift-invariant filters and design procedures. Shift-invariance is a graph signal processing concept proposed in Sandryhaila and Moura [33], and it is used here to impart the statistical properties of the graph onto filters. The design complexity of shift-invariant filters is shown to be linear. Section 5 defines shift-invariant filters which have constant learning complexity. Functional calculus is used to define filters which are functions of a given graph-shift operator. In Section 6, the previous classes of filters are discussed for the case in which the graph is undirected.

Table 1 Summary of filter design results for $\mathsf{A} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z} \times \mathscr{V}\right)\right)$

Filter Type	Complexity	Design Parameters
Linear and time-invariant	$\mathscr{O}\left(n^2\right)$	$\left\{\hat{a}_{j,k} \in L^\infty\left([0,1]\right)\right\}_{j,k\in\mathscr{V}}$
Linear and shift-invariant	$\mathscr{O}\left(n\right)$	$\left\{\hat{a}_{k} \in L^\infty\left([0,1]\right)\right\}_{k\in\mathscr{V}}$
Function of graph-shift operator	$\mathscr{O}\left(1\right)$	$\phi : U \to \mathbb{C}$, holomorphic

2 Preliminaries

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a graph with nodes $\mathscr{V} = \{0, \dots, n-1\}$ such that $n = |\mathscr{V}| < \infty$ and $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$. A weight function assigns a relationship between any two nodes with an edge connecting them $w : \mathscr{E} \to \mathbb{R}$. The function, w, defines the entries of the adjacency, or weight matrix, W ($[\mathsf{W}]_{j,k} = w(j,k)$). If w is symmetric (i.e. $w(j,k) = w(k,j)$), then the graph is undirected, otherwise it is directed. The degree of each node follows from the definition of W, $d_i = \sum_{j\in\mathscr{V}} w(i,j)$, and the diagonal degree matrix is $\mathsf{D} = diag(d_0, \dots, d_{n-1})$. The Laplacian of $\mathscr{G}$ is $\mathsf{L} = \mathsf{D} - \mathsf{W}$. If the weight matrix is undirected, then the Laplacian is positive semi-definite. Other matrices of interest are the normalized Laplacian, $\mathsf{L}_n = \mathsf{D}^{-1/2}\mathsf{L}\mathsf{D}^{-1/2}$, and the random walk Laplacian, $\mathsf{L}_r = \mathsf{D}^{-1}\mathsf{L}$.

We consider time-varying graph signals which can be represented as functions of time supported on the nodes of a fixed graph. This can be incorporated into the above graph notation by considering an extended graph $\tilde{\mathscr{G}} = (\tilde{\mathscr{V}}, \tilde{\mathscr{E}})$ where $\tilde{\mathscr{V}} = \mathbb{Z} \times \mathscr{V}$ and $\tilde{\mathscr{E}} \subseteq \tilde{\mathscr{V}} \times \tilde{\mathscr{V}}$. An example extended graph is depicted in Figure 1.

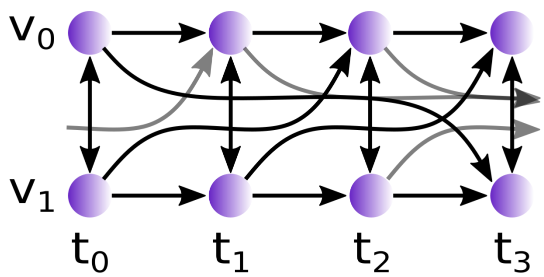

Fig. 1 An example extended graph $\tilde{\mathscr{G}}$. Each time $t \in \mathbb{Z}$ is associated with the node set $\mathscr{V} = \{v_0, v_1\}$ so that each node is indexed by time and space, (v,t). The extended graph admits edges between any two nodes regardless of proximity in time and space.

Thus, time-varying graph signals are functions on $\tilde{\mathscr{G}}$, equivalently vector-valued sequences indexed by time $\{\mathbf{x}[t] \in \mathbb{C}^n\}_{t\in\mathbb{Z}}$ or scalar functions on $\tilde{\mathscr{V}}$, $x : \mathbb{Z} \times \mathscr{V} \to \mathbb{C}$. We use both notations, the vector-valued sequence and scalar-valued function, as is convenient in context. Some knowledge of function spaces is assumed, namely the definitions and norms of $\ell^p(\mathbb{Z})$ and $L^p([0,1])$ for $1 \leq p \leq \infty$. The Lebesgue measure is used throughout on L^p-spaces. Some of the analysis will take place in finite dimensions on a function space $\ell^2(\mathscr{V})$ which is isomorphic to $\mathbb{C}^n$ with the usual Euclidean norm, but the primary analysis will occur in two Hilbert spaces, $\ell^2(\mathbb{Z} \times \mathscr{V})$ and $L^2([0,1] \times \mathscr{V})$, the time and frequency domain of time-varying graph signals.

Definition 1. The $\ell^2(\mathbb{Z} \times \mathscr{V})$-norm is defined as

$$\|x\|_{\ell^2(\mathbb{Z}\times\mathscr{V})} = \left(\sum_{t\in\mathbb{Z}}\sum_{v\in\mathscr{V}} |x(t,v)|^2\right)^{1/2} = \left(\sum_{t\in\mathbb{Z}} \|\mathbf{x}[t]\|^2_{\ell^2(\mathscr{V})}\right)^{1/2},$$

for a function $x : \mathbb{Z} \times \mathscr{V} \to \mathbb{C}$, and it defines the function space

$$\ell^2(\mathbb{Z} \times \mathscr{V}) = \left\{x : \mathbb{Z} \times \mathscr{V} \to \mathbb{C} \mid \|x\|_{\ell^2(\mathbb{Z}\times\mathscr{V})} < \infty\right\}.$$

Moreover, $\|\cdot\|_{\ell^2(\mathbb{Z}\times\mathscr{V})}$ is induced by an inner product

$$\langle x, y\rangle_{\ell^2(\mathbb{Z}\times\mathscr{V})} = \sum_{t\in\mathbb{Z}}\sum_{v\in\mathscr{V}} x(t,v)\bar{y}(t,v) = \sum_{t\in\mathbb{Z}} \langle \mathbf{x}[t], \mathbf{y}[t]\rangle_{\ell^2(\mathscr{V})}$$

where $x, y \in \ell^2(\mathbb{Z} \times \mathscr{V})$.[1]

Definition 2. The $L^2([0,1] \times \mathscr{V})$-norm is defined as

$$\|x\|_{L^2([0,1]\times\mathscr{V})} = \left(\int_{[0,1]} \sum_{v\in\mathscr{V}} |x(\omega,v)|^2 \, d\omega \right)^{1/2} = \left(\int_{[0,1]} \|\mathbf{x}(\omega)\|^2_{\ell^2(\mathscr{V})} \, d\omega \right)^{1/2}$$

for $x : [0,1] \times \mathscr{V} \to \mathbb{C}$ a measurable function and $d\omega$ is the Lebesgue measure. It defines the function space

$$L^2([0,1] \times \mathscr{V}) = \left\{ x : [0,1] \times \mathscr{V} \to \mathbb{C} \mid \|x\|_{L^2([0,1]\times\mathscr{V})} < \infty \right\}.$$

Moreover, $\|\cdot\|_{L^2([0,1]\times\mathscr{V})}$ is induced by an inner product

$$\langle x, y \rangle_{L^2([0,1]\times\mathscr{V})} = \int_{[0,1]} \sum_{v\in\mathscr{V}} x(\omega,v)\bar{y}(\omega,v) d\omega = \int_{[0,1]} \langle \mathbf{x}[t], \mathbf{y}[t] \rangle_{\ell^2(\mathscr{V})} \, d\omega$$

where $x, y \in L^2([0,1] \times \mathscr{V})$.

Fourier analysis plays a central role in signal processing of time-series signals, and it will play an important role in the design and analysis of filters on $\ell^2(\mathbb{Z} \times \mathscr{V})$. Fundamental results from Fourier analysis on $\ell^2(\mathbb{Z})$ and $L^2([0,1])$ for $1 \le p \le \infty$, including Plancherel's theorem and the convolution theorem carry over to $\ell^2(\mathbb{Z} \times \mathscr{V})$ and $L^2([0,1] \times \mathscr{V})$ with the following definitions of the Fourier and inverse Fourier transform (see e.g. [20] for reference).

Definition 3. The *Fourier transform* $\mathscr{F}$ of $\mathbf{x} \in \ell^2(\mathbb{Z} \times \mathscr{V})$ is defined as

$$(\mathscr{F}x)(\omega, v) = \hat{x}(\omega, v) = \sum_{t\in\mathbb{Z}} \mathrm{e}^{2\pi i\omega t} x(t, v),$$

where $\omega \in [0,1]$.

Definition 4. The *inverse Fourier transform* $\mathscr{F}^*$ of $\mathbf{x} \in L^2([0,1] \times \mathscr{V})$ is defined as

$$(\mathscr{F}^* x)(t, v) = \int_{[0,1]} \mathrm{e}^{-2\pi i\omega t} x(\omega, v) d\omega,$$

where $t \in \mathbb{Z}$ and $d\omega$ is the Lebesgue measure.

In the discrete signal processing setting, filtering builds on linear system theory and finite-dimensional linear algebra. These concepts need to be generalized for signals in $\ell^2(\mathbb{Z} \times \mathscr{V})$.

Definition 5. The *operator norm* of $\mathsf{A} : \ell^2(\mathbb{Z} \times \mathscr{V}) \to \ell^2(\mathbb{Z} \times \mathscr{V})$ is defined as

[1] $\bar{y}$ denotes the complex conjugate of y.

$$\|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} = \sup_{\mathbf{x}\in\ell^2(\mathbb{Z}\times\mathscr{V})} \frac{\|\mathsf{A}\mathbf{x}\|_{\ell^2(\mathbb{Z}\times\mathscr{V})}}{\|\mathbf{x}\|_{\ell^2(\mathbb{Z}\times\mathscr{V})}} = \sup_{\|\mathbf{x}\|_{\ell^2(\mathbb{Z}\times\mathscr{V})}=1} \|\mathsf{A}\mathbf{x}\|_{\ell^2(\mathbb{Z}\times\mathscr{V})}.$$

A is said to be *bounded* if $\|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} < \infty$. The set of bounded linear transformations is denoted $\mathscr{B}\left(\ell^2\left(\mathbb{Z}\times\mathscr{V}\right)\right)$.

Bounded operators are more amenable to analysis than unbounded operators in general, and for practical filtering applications, it is presumed that filtered graph signals should be bounded. Moreover, the set of bounded linear transformation $\mathscr{B}\left(\ell^2\left(\mathbb{Z}\times\mathscr{V}\right)\right)$ with identity E and composition is a Banach algebra [5].

Definition 6. For $\mathsf{A} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z}\times\mathscr{V}\right)\right)$ and $\mathbf{x},\mathbf{y} \in \ell^2\left(\mathbb{Z}\times\mathscr{V}\right)$, the *adjoint*, $\mathsf{A}^* \in \mathscr{B}\left(\ell^2\left(\mathbb{Z}\times\mathscr{V}\right)\right)$ satisfies

$$\langle \mathsf{A}\mathbf{x},\mathbf{y}\rangle_{\ell^2(\mathbb{Z}\times\mathscr{V})} = \langle \mathbf{x},\mathsf{A}^*\mathbf{y}\rangle_{\ell^2(\mathbb{Z}\times\mathscr{V})}.$$

An operator $\mathsf{A} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z}\times\mathscr{V}\right)\right)$ is said to to be *self-adjoint* if $\mathsf{A} = \mathsf{A}^*$.

There is a natural generalization of finite-dimensional linear operators to infinite-dimensional multiplication operators. Let $\mathbf{x} \in \ell^2\left(\mathbb{Z}\times\mathscr{V}\right)$, then for each $\mathsf{A} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z}\times\mathscr{V}\right)\right)$, there exists a unique kernel $\mathsf{K} : \mathbb{Z}\times\mathbb{Z} \to \mathscr{B}\left(\ell^2\left(\mathscr{V}\right)\right)$ such that

$$\left(\mathsf{A}\mathbf{x}\right)[t] = \lim_{N\to\infty}\left(\sum_{s=-N}^{N} \mathsf{K}(t,s)\mathbf{x}[s]\right). \tag{1}$$

Note that K is a finite-dimensional linear operator, and recall that $\ell^2\left(\mathscr{V}\right)$ is isomorphic to $\mathbb{C}^n$. It follows that K acts like a matrix on $\mathbb{C}^n$. This intuition then informs the matrix-vector representation of the action of A, a bi-infinite block matrix acting on a bi-infinite vector.

$$\mathsf{A}\mathbf{x} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \mathsf{K}(-1,-1) & \mathsf{K}(-1,0) & \mathsf{K}(-1,1) & \cdots \\ \cdots & \mathsf{K}(0,-1) & \mathsf{K}(0,0) & \mathsf{K}(0,1) & \cdots \\ \cdots & \mathsf{K}(1,-1) & \mathsf{K}(1,0) & \mathsf{K}(1,1) & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \mathbf{x}[-1] \\ \mathbf{x}[0] \\ \mathbf{x}[1] \\ \vdots \end{bmatrix} \tag{2}$$

In linear systems theory, matrix decompositions and the spectra of matrices play an important role. The canonical decomposition in finite dimensions is the Jordan spectral representation which follows (for a proof of the theorem, see e.g. [17]).

Theorem 1 (Jordan spectral representation). *Let* $\mathsf{A} \in \mathscr{B}\left(\ell^2\left(\mathscr{V}\right)\right)$. *Then, there exists* $m \le n = |\mathscr{V}|$ *distinct eigenvalues* $\left\{\lambda_k \in \mathbb{C}\right\}_{1\le k\le m}$, *projections* $\left\{\mathsf{P}_k \in \mathscr{B}\left(\ell^2\left(\mathscr{V}\right)\right)\right\}_{1\le k\le m}$, *and nilpotents* $\left\{\mathsf{N}_k \in \mathscr{B}\left(\ell^2\left(\mathscr{V}\right)\right)\right\}_{1\le k\le m}$ *such that*

$$\mathsf{A} = \sum_{k=0}^{m-1} \lambda_k \mathsf{P}_k + \mathsf{N}_k \tag{3}$$

with the following properties:

1. $\mathsf{P}_j\mathsf{P}_k = \mathsf{P}_k\mathsf{P}_j = \delta_{jk}\mathsf{P}_k$
2. $\mathsf{P}_k\mathsf{N}_k\mathsf{P}_k = \mathsf{N}_k$
3. $(\mathsf{N}_k)^n = 0$
4. $\sum_{k=0}^{m-1}\mathsf{P}_k = \mathsf{E}$ *(the identity operator on* $\ell^2(\mathscr{V})$*).*

Properties 1–4 of Theorem 1 imply additionally that

$$\mathsf{P}_k\mathsf{N}_k = \mathsf{N}_k\mathsf{P}_k = \mathsf{N}_k \tag{4}$$

and

$$\mathsf{P}_j\mathsf{N}_k = \mathsf{N}_k\mathsf{P}_j = \delta_{jk}\mathsf{N}_k. \tag{5}$$

The Jordan spectral representation is perhaps less familiar than the equivalent Jordan normal form, but the projections and nilpotents will be important for the analysis of Sections 4 and 5.

In the following, filter design will rely heavily on spectral theory which extends the notion of matrix decomposition to infinite-dimensional linear operators on Banach spaces. This theory deals with the spectrum of an operator in place of the eigenvalues of a matrix. Some pertinent definitions follow.

Definition 7. The *resolvent set* of $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)$ is

$$\rho(\mathsf{A}) = \left\{z \in \mathbb{C} \mid (\mathsf{A} - z\mathsf{E})^{-1} \in \mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)\right\}.$$

where E denotes the identity operator on $\ell^2(\mathbb{Z}\times\mathscr{V})$.

Definition 8. The *spectrum* of $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)$ is $\sigma(\mathsf{A}) = \mathbb{C}\setminus\rho(\mathsf{A})$.

Definition 9. The *resolvent* of $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)$ is the operator-valued function $\mathsf{R} : \rho(\mathsf{A}) \to \mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)$,

$$\mathsf{R}_{\mathsf{A}}(z) = (A - z\mathsf{E})^{-1}.$$

The relationship between the spectrum of an operator and an eigenvalue of a matrix can be seen from the above definitions. The spectrum is the subset of $\mathbb{C}$ such that $\mathsf{A} - z\mathsf{E}$ is not bijective, equivalent to an eigenvalue in finite dimensions. The spectrum of a bounded operator is closed, bounded, and never empty [5], and the resolvent set is everything except the spectrum. As opposed to the eigenvalues of a matrix, which are a finite set of elements in $\mathbb{C}$, the spectrum of a bounded operator in general has in addition to a point spectrum, a continuous and a residual spectrum [5]. As the eigenvalues of a matrix are a special case of the spectrum of a bounded operator, the spectrum of $\mathsf{A} \in \ell^2(V)$ will also be denoted $\sigma(\mathsf{A})$.

3 Linear and Time-Invariant Filters on $\ell^2(\mathbb{Z}\times\mathscr{V})$

Linear, time-invariant systems theory underlies much of signal processing. With respect to filtering operations, linear time-invariant filters are amenable to analysis and yield well-understood behavior. As discussed in Section 2, operators in $\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)$ naturally generalize matrices. It remains only to extend the notion of time-invariance.

3.1 Definition of Time-Invariance in $\ell^2(\mathbb{Z}\times\mathscr{V})$

Generalizations of time-invariance exist for signal processing on spatial data (e.g. translation invariance in image processing), but this chapter concerns graph signals with both spatial and time dimensions. Therefore, time-invariance in $\ell^2(\mathbb{Z}\times\mathscr{V})$ should simply extend the definition of time-invariance from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z}\times\mathscr{V})$. Time-invariant filters by convention means that the entries of the matrix are not functions of time. Then, there is a certain invariance to when the filter is applied. If this definition is deconstructed, it depends first on a notion of time-evolution and second on an invariant action of the operator.

Let time-evolution of a discrete signal $x\in\ell^2(\mathbb{Z})$ be associated with the time-shift operator, $\mathsf{T}\in\mathscr{B}\left(\ell^2(\mathbb{Z})\right)$,

$$(\mathsf{T}\mathbf{x})[t]=\mathbf{x}[t-1]. \tag{6}$$

For finite-dimensional signals, T is a circulant matrix with ones on the sub-diagonal, and it is diagonalized by the discrete Fourier transform [33]. The same operation can be defined for time-varying graph signals for $\mathsf{T}\in\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)$ and $\mathbf{x}\in\ell^2(\mathbb{Z}\times\mathscr{V})$, which acts like the shift operator except for being infinitely block circulant with the multiplicative identity $\mathsf{E}\in\ell^2(\mathscr{V})$ on the sub-diagonal,

$$\mathsf{T}=\begin{bmatrix}\ddots & & & & \\ & \mathsf{E} & & & \\ & & \mathsf{E} & & \\ & & & \mathsf{E} & \\ & & & & \ddots\end{bmatrix}. \tag{7}$$

Note that for for $\mathsf{T}\in\mathscr{B}\left(\ell^2(\mathbb{Z})\right)$, $\mathsf{E}=1$. Thus, the time-shift operation advances not just a scalar signal but the scalar graph signal one step in time.

Consider the action of a linear filter $\mathsf{A}\in\mathscr{B}\left(\ell^2(\mathbb{Z})\right)$ on $x\in\ell^2(\mathbb{Z})$. When we say time-invariant filtering, we mean

$$(\mathsf{AT}x)[t]=(\mathsf{TA}x)[t], \tag{8}$$

which is also understood as covariance (or equivariance) to time-shift. If $\mathbf{x} \in \ell^2(\mathbb{Z} \times \mathscr{V})$ and $\mathsf{T} \in \mathscr{B}\left(\ell^2(\mathbb{Z} \times \mathscr{V})\right)$, then the same condition should define time-invariance for graph signals. Thus, for a filter $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z} \times \mathscr{V})\right)$, it must commute with the cyclic group generated by the time shift operator T, $\mathscr{T} = \langle \mathsf{T} \rangle$.

Definition 10. $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z} \times \mathscr{V})\right)$ is called *time-invariant* if A is in the commutant of $\mathscr{T}$, $\left\{\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z} \times \mathscr{V})\right) \mid \mathsf{A}\mathscr{T} = \mathscr{T}\mathsf{A}\right\}$.

3.2 Laurent Operators on $\ell^2(\mathbb{Z} \times \mathscr{V})$

In finite dimensions, Toeplitz matrices commute with the circulant shift matrix because both are diagonalized by the discrete Fourier transform. These matrices define scalar convolutions. The matrix representation of Eq. (2) makes possible the generalization of Toeplitz operators, and also convolution, to operators on $\ell^2(\mathbb{Z} \times \mathscr{V})$.

Definition 11. Let $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z} \times \mathscr{V})\right)$ with kernel $\mathsf{K} : \mathbb{Z} \times \mathbb{Z} \to \mathscr{B}\left(\ell^2(\mathscr{V})\right)$. Then, A is said to be *Laurent* if $\mathsf{K}(s,t) = \mathsf{K}(s+d, t+d)$ for all $d \in \mathbb{Z}$.

Denote $\mathsf{K}_s = \mathsf{K}(s,0)$. Laurent operators generalize convolution operators as can be seen from the action of a Laurent operator,

$$(\mathsf{A}\mathbf{x})[t] = \lim_{N\to\infty} \left(\sum_{t=-N}^{N} \mathsf{K}_{t-s} \mathbf{x}[s] \right), \tag{9}$$

which has a block Toeplitz matrix representation,

$$\mathsf{A}\mathbf{x} = \begin{bmatrix} \ddots & \ddots & \ddots & & & & \\ \cdots & \mathsf{K}_1 & \mathsf{K}_0 & \mathsf{K}_{-1} & \cdots & & \\ & \cdots & \mathsf{K}_1 & \mathsf{K}_0 & \mathsf{K}_{-1} & \cdots & \\ & & \cdots & \mathsf{K}_1 & \mathsf{K}_0 & \mathsf{K}_{-1} & \cdots \\ & & & \ddots & \ddots & \ddots & \end{bmatrix} \begin{bmatrix} \vdots \\ \mathbf{x}[-1] \\ \mathbf{x}[0] \\ \mathbf{x}[1] \\ \vdots \end{bmatrix}. \tag{10}$$

Time-invariant filters are inherently Laurent, and vice-versa, as the following theorem shows.

Theorem 2. $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z} \times \mathscr{V})\right)$ *is time-invariant if and only if* A *is Laurent.*

Proof. $\Leftarrow$
First, let A be Laurent. It is sufficient to show that $\mathsf{A}\mathsf{T}^d = \mathsf{T}^d\mathsf{A}$ for some $d \in \mathbb{Z}$.

$$\begin{aligned}\left(\mathsf{A}\mathsf{T}^d\mathbf{x}\right)[t] &= \lim_{N\to\infty}\sum_{s=-N}^{N}\mathsf{K}_{t-s}\left(\mathsf{T}^d\mathbf{x}\right)[s]\\ &= \lim_{N\to\infty}\sum_{s=-N}^{N}\mathsf{K}_{t-s}\mathbf{x}[s-d]\\ &= \lim_{N\to\infty}\sum_{s'=-N}^{N}\mathsf{K}_{t-(s'+d)}\mathbf{x}[s']\\ &= \mathsf{T}^d\left(\lim_{N\to\infty}\sum_{s'=-N}^{N}\mathsf{K}_{t-s'}\mathbf{x}[s']\right)\\ &= \left(\mathsf{T}^d\mathsf{A}\mathbf{x}\right)[t]\end{aligned}$$

$\Rightarrow$

Now, let A be time-invariant. By Def. 10, A commutes with $\mathscr{T}$. Without loss of generality, choose $\mathsf{T}^d \in \mathscr{T}$ for some $d \in \mathbb{Z}$. It is necessary to show that $\mathsf{K}(t,s) = \mathsf{K}(t+d,s+d)$. Let $\mathbf{x} \in \ell^2(\mathbb{Z}\times\mathscr{V})$.

$$\begin{aligned}\left(\mathsf{A}\mathsf{T}^d\mathbf{x}\right)[t+d] &= \left(\mathsf{T}^d\mathsf{A}\mathbf{x}\right)[t+d]\\ \lim_{N\to\infty}\sum_{s=-N}^{N}\mathsf{K}(t+d,s)\left(\mathsf{T}^d\mathbf{x}\right)[s] &= \mathsf{T}^d\left(\lim_{N\to\infty}\sum_{s=-N}^{N}\mathsf{K}(t+d,s)\mathbf{x}[s]\right)\\ \lim_{N\to\infty}\sum_{s=-N}^{N}\mathsf{K}(t+d,s)\mathbf{x}[s-d] &= \lim_{N\to\infty}\sum_{s=-N}^{N}\mathsf{K}(t,s)\mathbf{x}[s]\\ \lim_{N\to\infty}\sum_{s'=-N}^{N}\mathsf{K}(t+d,s'+d)\mathbf{x}[s'] &= \lim_{N\to\infty}\sum_{s=-N}^{N}\mathsf{K}(t,s)\mathbf{x}[s]\end{aligned}$$

By the uniqueness of K, A is Laurent. □

3.3 Design of Linear and Time-Invariant Filters on $\ell^2(\mathbb{Z}\times\mathscr{V})$

Next, we exploit Theorem 2 to develop a filter design procedure. Consider the following.

Theorem 3. *Let* $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)$ *be Laurent with* $\sum_{t\in\mathbb{Z}}\|\mathsf{K}_t\|_{\mathscr{B}(\ell^2(\mathscr{V}))} < \infty$*. Then,* $\sigma(\mathsf{A}) = \cup_{\omega\in[0,1]}\sigma\left(\hat{\mathsf{A}}(\omega)\right)$ *and*

$$(\mathscr{F}\mathsf{A}\mathbf{x})(\omega) = \hat{\mathsf{A}}(\omega)\hat{\mathbf{x}}(\omega) \tag{11}$$

for $\mathbf{x} \in \ell^2(\mathbb{Z}\times\mathscr{V})$ *where*

$$\hat{\mathsf{A}}(\omega) = \sum_{t\in\mathbb{Z}} \mathrm{e}^{2\pi i\omega t}\mathsf{K}_t. \tag{12}$$

Moreover,

$$\|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} = \max_{\omega\in[0,1]} s_{\max}\left(\hat{\mathsf{A}}(\omega)\right) \tag{13}$$

where $s_{\max}(\cdot)$ is the maximum singular value.[2]

Proof. Let $\mathbf{x} \in \ell^2(\mathbb{Z}\times\mathscr{V})$.

$$\begin{aligned}
(\mathscr{F}\mathsf{A}\mathbf{x})(\omega) &= \sum_{t\in\mathbb{Z}} \mathrm{e}^{2\pi i\omega t}\left(\lim_{N\to\infty}\left(\sum_{s=-N}^{N}\mathsf{K}_{t-s}\mathbf{x}[s]\right)\right)\\
&= \lim_{N\to\infty}\sum_{t\in\mathbb{Z}}\sum_{s=-N}^{N}\mathrm{e}^{2\pi i\omega(t-s)}\mathsf{K}_{t-s}\mathrm{e}^{2\pi i\omega s}\mathbf{x}[s]\\
&= \left(\lim_{N\to\infty}\sum_{t'=-N}^{N}\mathrm{e}^{2\pi i\omega t'}\mathsf{K}_{t'}\right)\left(\sum_{s\in\mathbb{Z}}\mathrm{e}^{2\pi i\omega s}\mathbf{x}[s]\right)\\
&= \left(\lim_{N\to\infty}\sum_{t'=-N}^{N}\mathrm{e}^{2\pi i\omega t'\mathsf{K}_{t'}}\right)(\mathscr{F}\mathbf{x})(\omega)
\end{aligned}$$

The limit exists by absolute summability of the kernels. Moreover, $\hat{\mathsf{A}}$ is norm-continuous for $\omega\in[0,1]$. This leads to the convention that $\hat{A}$ is diagonalized by the Fourier transform, $\mathsf{A} = \mathscr{F}^*\hat{A}\mathscr{F}$. The spectrum identity is established next. It follows that

$$\mathsf{A} - z\mathsf{E} = \mathscr{F}^*\left(\hat{\mathsf{A}}(\omega) - z\mathsf{E}\right)\mathscr{F}.$$

If $z\in\rho(\mathsf{A})$, then there exists a $\mathsf{B}\in\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)$ such that $(\mathsf{A}-z)\mathsf{B} = \mathsf{E}$.

$$\begin{aligned}
(\mathsf{A}-z\mathsf{E})\,\mathsf{B} &= \mathsf{E}\\
\mathscr{F}^*\left(\hat{\mathsf{A}}(\omega)-z\mathsf{E}\right)\mathscr{F}\mathsf{B} &= \mathsf{E}\\
\left(\hat{\mathsf{A}}(\omega)-z\mathsf{E}\right)\hat{\mathsf{B}}(\omega) &= \mathsf{E}
\end{aligned}$$

Let $S = \cup_{\omega\in[0,1]}\sigma\left(\hat{\mathsf{A}}(\omega)\right)$. For $z\notin S$, $\hat{\mathsf{A}}(\omega)-z\mathsf{E}$ is invertible on $\ell^2(\mathscr{V})$, and for $z\in S$, $\hat{\mathsf{A}}(\omega)-z\mathsf{E}$ is not invertible, $S\subseteq\sigma(\mathsf{A})\subseteq S$. Thus, $\sigma(\mathsf{A}) = S$. Lastly, the norm identity is proved.

---

[2] $\sigma(\cdot)$ here denotes the spectrum of an infinite-dimensional linear operator and the spectrum (eigenvalues) of a finite-dimensional matrix.

$$
\begin{aligned}
\|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} &= \left\|\hat{\mathsf{A}}\right\|_{\mathscr{B}\left(L^2([0,1]\times\mathscr{V})\right)} \\
&= \sup_{\|\hat{\mathbf{x}}\|_{L^2([0,1]\times\mathscr{V})}=1} \left\|\hat{\mathsf{A}}\hat{\mathbf{x}}\right\|_{\ell^2(\mathbb{Z}\times\mathscr{V})} \\
&\leq \sup_{\|\hat{\mathbf{x}}\|_{L^2([0,1]\times\mathscr{V})}=1} \operatorname*{ess\,sup}_{\omega\in[0,1]} \left\|\hat{\mathsf{A}}(\omega)\hat{\mathbf{x}}(\omega)\right\|_{\mathscr{B}\left(\ell^2(\mathscr{V})\right)} \\
&\leq \operatorname*{ess\,sup}_{\omega\in[0,1]} \sup_{\|\hat{\mathbf{x}}\|_{L^2([0,1]\times\mathscr{V})}=1} \left\|\hat{\mathsf{A}}(\omega)\hat{\mathbf{x}}(\omega)\right\|_{\mathscr{B}\left(\ell^2(\mathscr{V})\right)} \\
&= \operatorname*{ess\,sup}_{\omega\in[0,1]} s_{\max}\left(\hat{\mathsf{A}}(\omega)\right)
\end{aligned}
$$

So far, this has only established an inequality, but the equality can be shown to be attained for a particular $\mathbf{x}\in\ell^2(\mathbb{Z}\times\mathscr{V})$. The goal is to find a unit norm function which lumps its mass on the measurable set at which $s_{\max}\left(\hat{\mathsf{A}}\right)$ attains its essential supremum while also being in the invariant subspace associated with the maximum singular value on that set. Let $\Omega\subset[0,1]$ be the set on which $s_{\max}\left(\hat{\mathsf{A}}\right)$ achieves its essential supremum. Then, $\hat{\mathsf{A}}^*(\Omega)\hat{\mathsf{A}}(\Omega)$ has an eigenvector $\mathbf{u}(\omega)$ for $\omega\in\Omega$. $\mathbf{u}$ will certainly not be a continuous function of $\omega\in[0,1]$, but it is possible to find a sequence of continuous functions $\{\mathbf{x}_k\}_{k\in\mathbb{N}}$ which converge to $\mathbf{u}$ on Ω and are zero otherwise, for which the equality is achieved. Lastly, since $[0,1]$ is a closed set and $\hat{\mathsf{A}}$ is norm-continuous for $\omega\in[0,1]$, $s_{\max}\left(\hat{\mathsf{A}}(\omega)\right)$ is attained on $[0,1]$.

$$
\operatorname*{ess\,sup}_{\omega\in[0,1]} s_{\max}\left(\hat{\mathsf{A}}(\omega)\right) = \max_{\omega\in[0,1]} s_{\max}\left(\hat{\mathsf{A}}(\omega)\right)
$$

□

As a result of Theorem 3, linear and time-invariant operators on $\ell^2(\mathbb{Z}\times\mathscr{V})$ are Laurent, these operators A have a spectral form given by Eq. 12, where $\hat{\mathsf{A}}:[0,1]\to\ell^2(\mathscr{V})$ is a finite-dimensional operator-valued function of $\omega\in[0,1]$. The action of A on a signal $\mathbf{x}\in\ell^2(\mathbb{Z}\times\mathscr{V})$ is

$$
(\mathsf{A}\mathbf{x})[t] = \left(\mathscr{F}^*\left[\hat{\mathsf{A}}(\omega)\hat{\mathbf{x}}(\omega)\right]\right)[t] \tag{14}
$$

where $\hat{\mathbf{x}}\in L^2([0,1]\times\mathscr{V})$. This action also admits a matrix-vector representation

$$
(\mathsf{A}\mathbf{x})[t] = \left(\mathscr{F}^*\left\{\begin{bmatrix}\hat{a}_{0,0}(\omega) & \cdots & \hat{a}_{0,n-1}(\omega)\\ \vdots & \ddots & \vdots\\ \hat{a}_{n-1,0}(\omega) & \cdots & \hat{a}_{n-1,n-1}(\omega)\end{bmatrix}\begin{bmatrix}\hat{x}_0(\omega)\\ \vdots\\ \hat{x}_{n-1}(\omega)\end{bmatrix}\right\}\right)[t]. \tag{15}
$$

Here, $\hat{\mathsf{A}}$ acts as a transfer function on a vector space $\ell^2(\mathscr{V})$.

The significance of Theorem 2 and Eq. (14) is that the design of linear and time-invariant filters can be carried out in the frequency domain where the action of the operator is a matrix-vector product. Equation (15) shows that there are $\mathscr{O}\left(n^2\right)$ parameters in the filter, of which each is a function $\left\{\hat{a}_{j,k}:[0,1]\to\mathbb{C}\right\}_{j,k\in\mathscr{V}}$. In fact, a stronger statement can be made about the functions $\hat{a}_{j,k}$, that they must be bounded.

Theorem 4. *Let $\hat{\mathsf{A}} : [0,1] \to \mathscr{B}\left(\ell^2(\mathscr{V})\right)$ be a a finite-dimensional operator-valued measurable function with entries $\hat{a}_{j,k} : [0,1] \to \mathbb{C}$ for $j,k \in \mathscr{V}$. Then, $\mathsf{A} = \mathscr{F}^* \hat{\mathsf{A}} \mathscr{F}$ defines a Laurent operator, and $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z} \times \mathscr{V})\right)$ if and only if $\hat{a}_{j,k} \in L^\infty([0,1])$ for $j,k \in \mathscr{V}$. Furthermore,*

$$\max_{j,k\in\mathscr{V}} \left\|\hat{a}_{j,k}\right\|_{L^\infty([0,1])} \leq \|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} \leq \left(\sum_{j\in\mathscr{V}} \sum_{j\in\mathscr{V}} \left\|\hat{a}_{j,k}\right\|_{L^\infty([0,1])}\right)^{1/2}. \tag{16}$$

Proof. By Theorem 2, $\mathsf{A} = \mathscr{F}^* \hat{\mathsf{A}} \mathscr{F}$ is Laurent. Eq. (16) will be proved next. From there, the remainder of the theorem follows easily. It is known that for $\mathsf{A} \in \ell^2(\mathscr{V})$, $\max_{j,k\in\mathscr{V}} \left|a_{j,k}\right| \leq s_{\max}(\mathsf{A}) \leq \|\mathsf{A}\|_F$ (see e.g. [12]). Now, consider $\mathsf{A} = \mathscr{F}^* \hat{\mathsf{A}} \mathscr{F}$.[3]

$$\begin{aligned}
\max_{j,k\in\mathscr{V}} \|\hat{a}\|^2_{\ell^\infty([0,1])} &= \operatorname*{ess\,sup}_{\omega\in[0,1]} \max_{j,k\in\mathscr{V}} \left|\hat{a}_{j,k}(\omega)\right|^2 \\
&\leq \operatorname*{ess\,sup}_{\omega\in[0,1]} s_{\max}\left(\hat{\mathsf{A}}(\omega)\right) \\
&= \|\mathsf{A}\|^2_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} \\
&= \operatorname*{ess\,sup}_{\omega\in[0,1]} s_{\max}\left(\hat{\mathsf{A}}(\omega)\right) \\
&\leq \operatorname*{ess\,sup}_{\omega\in[0,1]} \sum_{j\in\mathscr{V}} \sum_{k\in\mathscr{V}} \left|\hat{a}_{j,k}(\omega)\right|^2 \leq \sum_{j\in\mathscr{V}} \sum_{k\in\mathscr{V}} \left\|\hat{a}_{j,k}\right\|^2_{\ell^\infty([0,1])}
\end{aligned}$$

This establishes Eq. (16). Now, if $\hat{a}_{j,k} \in L^\infty([0,1])$ for $j,k \in \mathscr{V}$, then $\|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)}$ is bounded. □

3.4 Example

We aim to implement a bandpass filter on a signal $\mathbf{x} \in \ell^2(\mathbb{Z} \times \mathscr{V})$ using the framework of Section 3.3. A bandpass filter passes a signal at a specific range of frequencies and attenuates the signal at frequencies outside of the specified range. Implementing a bandpass filter in this construction looks very similar to a bandpass filter on a scalar time-series signal $\ell^2(\mathbb{Z})$. On $\ell^2(\mathbb{Z} \times \mathscr{V})$, the transfer function should be approximately equal to the multiplicative identity E for the desired range of frequencies and close to the multiplicative zero element $\mathsf{0}$ for those frequencies outside. Let a measurable subset $\Omega \subset [0,1]$ be the desired frequency range. Then, an ideal bandpass filter $\mathsf{A} = \mathscr{F}^* \mathsf{A} \mathscr{F}$ would be

[3] Unlike in Theorem 3, $\hat{\mathsf{A}} : [0,1] \to \ell^2(\mathscr{V})$ is not necessarily norm continuous. $\mathscr{F}^* : L^\infty([0,1]) \to \ell^1(\mathbb{Z})$ is not bijective, and K may not be absolutely summable. Thus, $\|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} = \operatorname{ess\,sup}_{\omega\in[0,1]} s_{\max}\left(\hat{\mathsf{A}}(\omega)\right)$.

$$\hat{\mathsf{A}}(\omega) = \begin{cases} \mathsf{E} & \omega \in \Omega \\ 0 & o.w. \end{cases}. \tag{17}$$

This would correspond to choosing $\hat{a}_{j,k}(\omega) = \delta_{j,k}\chi_\Omega(\omega)$ for all $j,k \in \mathscr{V}$, where

$$\chi_\Omega(\omega) = \begin{cases} 1 & \omega \in \Omega \\ 0 & o.w. \end{cases}. \tag{18}$$

Since $\chi_\Omega \in L^\infty([0,1])$, this defines a bounded operator $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z} \times \mathscr{V})\right)$ by Theorem 4. The action of A on $\mathbf{x}$ would be

$$(\mathsf{A}\mathbf{x})[t] = \int_\Omega \mathrm{e}^{-2\pi i \omega t} \hat{\mathbf{x}}(\omega) d\omega. \tag{19}$$

In the limit as $d\omega(\Omega) \to 0$ so that $\Omega = \{\omega_0\}$, this would implement an ideal band-pass,[4]

$$(\mathsf{A}\mathbf{x})[t] = \mathrm{e}^{-2\pi i \omega_0 t} \hat{\mathbf{x}}(\omega_0). \tag{20}$$

4 Linear and Shift-Invariant Filters on $\ell^2(\mathbb{Z} \times \mathscr{V})$

To this point, the graph geometry has not informed the design procedure. As argued in Sandryhaila and Moura [33], the weighted adjacency matrix describes the spatial evolution of a signal. It captures the flow rate between nodes or the conditional probabilities between random variables. These are relationships between distinctly spatial and non-temporal elements of the signal. A similar intuition follows from the random walk Laplacian as is claimed in [43], which propagates probability mass between the nodes through a Markov process. Requiring that linear and time-invariant filters additionally be invariant to this spatial evolution offers one way to incorporate the information from the underlying graph. This shift-invariance connotes a spatial meaning as opposed to the temporal one of time-invariance.

4.1 Definition of Shift-Invariance in $\ell^2(\mathbb{Z} \times \mathscr{V})$

This section will investigate the design of filters that are invariant to a spatial graph evolution that is also time-invariant. As time-invariance is defined in Section 3, shift-invariance can be defined as commuting with a given graph-shift operator. This section will consider an arbitrary graph-shift operator, $\mathsf{S} \in \mathscr{B}\left(\ell^2(\mathscr{V})\right)$ that could be an adjacency matrix or Laplacian and potentially be self-adjoint, unitary, sparse, or any other desired property. Then, shift-invariance would mean commuting with

[4] This converges only in the distribution sense as it amounts to $\hat{a}_{j,k}(\omega) \to \delta(\omega - \omega_0)$, a non-measurable function.

the cyclic group generated by S, $\mathscr{S} = \langle \mathsf{S} \rangle$. Since by Theorem 2, any linear and time-invariant filter must be Laurent, considering time-invariant graph-shift operators (i.e. Laurent) ensures that any shift-invariant filter is also time-invariant (i.e. Laurent). This section proceeds with the following definition for shift-invariance.

Definition 12. $\mathsf{A} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z} \times \mathscr{V}\right)\right)$ is called *shift-invariant* to $\mathsf{S} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z} \times \mathscr{V}\right)\right)$ Laurent if A is in the commutant of $\mathscr{S} = \langle \mathsf{S} \rangle$,

$$\left\{\mathsf{A} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z} \times \mathscr{V}\right)\right) \mid \mathsf{A}\mathscr{S} = \mathscr{S}\mathsf{A}\right\}.$$

Laurent graph-shift operators convey a special physical significance. If the graph-shift operator S represents the weighted adjacency matrix or random walk matrix, then the kernel K_{s-t} captures the weights or transition probabilities for nodes at a temporal distance $s-t$. That is $\left[\mathsf{K}_{s-t}\right]_{j,k}$ is the weight or transition probability from node v_j at time s to node v_k at time t, and because the kernel depends only on the temporal distance between nodes, S captures a special stationary process as in Fig. 1.

4.2 Theoretical Results on Shift-Invariant Filters in $\ell^2\left(\mathbb{Z} \times \mathscr{V}\right)$

The first results on the design of shift-invariant filters follow from standard results on simultaneous diagonalization of matrices [12].

Theorem 5. *Let* $\mathsf{A}, \mathsf{S} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z} \times \mathscr{V}\right)\right)$ *be Laurent operators with Jordan spectral representations given pointwise for a.e.* $\omega \in [0,1]$ *by*

$$\hat{\mathsf{S}}(\omega) = \sum_{k=0}^{m(\omega)} \lambda_k(\omega)\mathsf{P}_k(\omega) + \mathsf{N}_k(\omega) \tag{21}$$

and

$$\hat{\mathsf{A}}(\omega) = \sum_{j=0}^{p(\omega)} \nu_j(\omega)\mathsf{Q}_j(\omega) + \mathsf{M}_j(\omega) \tag{22}$$

respectively. Then, A *is shift-invariant with respect to* S *if and only if*

$$\mathsf{P}_k(\omega)\mathsf{Q}_j(\omega) = \mathsf{Q}_j(\omega)\mathsf{P}_k(\omega) \tag{23}$$
$$\mathsf{P}_k(\omega)\mathsf{M}_j(\omega) = \mathsf{M}_j(\omega)\mathsf{P}_k(\omega) \tag{24}$$
$$\mathsf{N}_k(\omega)\mathsf{Q}_j(\omega) = \mathsf{Q}_j(\omega)\mathsf{N}_k(\omega) \tag{25}$$
$$\mathsf{N}_k(\omega)\mathsf{M}_j(\omega) = \mathsf{M}_j(\omega)\mathsf{N}_k(\omega) \tag{26}$$

for all $k \in \{0,\ldots,m(\omega)\}$, $j \in \{0,\ldots,p(\omega)\}$, *and* $\omega \in [0,1]$.

Proof. It will help to begin with an intermediate result: if $\mathsf{A}, \mathsf{S} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z} \times \mathscr{V}\right)\right)$ are Laurent, then A commutes with $\mathscr{S}$ if and only if

$$\hat{\mathsf{A}}(\omega)\hat{\mathsf{S}}(\omega) = \hat{\mathsf{S}}(\omega)\hat{\mathsf{A}}(\omega) \tag{27}$$

almost everywhere for $\omega \in [0,1]$.

$\Leftarrow$

First, assume that Eq. (27) is true. Then,

$$\begin{aligned}
\hat{\mathsf{A}}(\omega)\hat{\mathsf{S}}(\omega)\hat{\mathbf{x}}(\omega) &= \hat{\mathsf{S}}(\omega)\hat{\mathsf{A}}(\omega)\hat{\mathbf{x}}(\omega)\\
\left(\mathscr{F}^*\hat{\mathsf{A}}(\omega)\hat{\mathsf{S}}(\omega)\hat{\mathbf{x}}(\omega)\right)[t] &= \left(\mathscr{F}^*\hat{\mathsf{S}}(\omega)\hat{\mathsf{A}}(\omega)\hat{\mathbf{x}}(\omega)\right)[t]\\
(\mathsf{AS}\mathbf{x})[t] &= (\mathsf{SA}\mathbf{x})[t]
\end{aligned}$$

It also follows from Eq. (27) that $\hat{\mathsf{A}}(\omega)\mathsf{S}^d(\omega) = \mathsf{S}^d(\omega)\hat{\mathsf{A}}(\omega)$ for any $d \in \mathbb{Z}$. From that, it can be shown that A commutes with any $\mathsf{S}^d \in \mathscr{S}$ by the same argument as for S.

$\Rightarrow$

Now, assume that A commutes with any $\mathsf{S}^d \in \mathscr{S}$.

$$\begin{aligned}
\left(\mathsf{AS}^d\mathbf{x}\right)[t] &= \left(\mathsf{S}^d\mathsf{A}\mathbf{x}\right)[t]\\
\mathscr{F}\left(\mathsf{AS}^d\mathbf{x}\right)[t] &= \mathscr{F}\left(\mathsf{S}^d\mathsf{A}\mathbf{x}\right)[t]\\
\hat{\mathsf{A}}(\omega)\hat{\mathsf{S}}^d(\omega)\hat{\mathbf{x}}(\omega) &= \hat{\mathsf{S}}^d(\omega)\hat{\mathsf{A}}(\omega)\hat{\mathbf{x}}(\omega)
\end{aligned}$$

The intermediate result is proven.

Now, it suffices to prove that $\hat{\mathsf{A}}(\omega)$ commutes with $\hat{\mathsf{S}}(\omega)$ almost everywhere on $\omega \in [0,1]$ if and only if Eqs. (23), (24), (25), and (26) hold for all $k \in \{0,\ldots,m(\omega)\}$, $j \in \{0,\ldots,p(\omega)\}$, and a.e. $\omega \in [0,1]$.

$\Leftarrow$

First, assume that the respective projections and nilpotents commute. To make it more readable, the dependence on ω is dropped, but it is to be understood that this condition must hold pointwise almost everyone for $\omega \in [0,1]$.

$$\begin{aligned}
\hat{\mathsf{A}}\hat{\mathsf{S}} &= \left(\sum_{j=0}^{p} \nu_j\mathsf{Q}_j + \mathsf{M}_j\right)\left(\sum_{k=0}^{m} \lambda_k\mathsf{P}_k + \mathsf{N}_k\right)\\
&= \sum_{j=0}^{p}\sum_{k=0}^{m} \nu_j\lambda_k\mathsf{Q}_j\mathsf{P}_k + \nu_j\mathsf{Q}_j\mathsf{N}_k + \lambda_k\mathsf{M}_j\mathsf{P}_k + \mathsf{M}_j\mathsf{N}_k\\
&= \sum_{j=0}^{p}\sum_{k=0}^{m} \nu_j\lambda_k\mathsf{P}_k\mathsf{Q}_j + \nu_j\mathsf{N}_k\mathsf{Q}_j + \lambda_k\mathsf{P}_k\mathsf{M}_j + \mathsf{N}_k\mathsf{M}_j\\
&= \left(\sum_{k=0}^{m} \lambda_k\mathsf{P}_k + \mathsf{N}_k\right)\left(\sum_{j=0}^{p} \nu_j\mathsf{Q}_j + \mathsf{M}_j\right)\\
&= \hat{\mathsf{S}}\hat{\mathsf{A}}
\end{aligned}$$

$\Rightarrow$

Now, assume that $\hat{\mathsf{A}}(\omega)$ and $\hat{\mathsf{S}}(\omega)$ commute almost everywhere on $\omega \in [0,1]$. Then, the resolvents commute: Let $z_1 \in \rho\left(\hat{\mathsf{A}}(\omega)\right)$ and $z_2 \in \rho\left(\hat{\mathsf{S}}(\omega)\right)$.

$$\begin{aligned}
\left(\hat{\mathsf{A}}(\omega)-z_1\mathsf{E}\right)\left(\hat{\mathsf{S}}(\omega)-z_2\mathsf{E}\right) &= \hat{\mathsf{A}}(\omega)\hat{\mathsf{S}}(\omega)-z_2\hat{\mathsf{A}}(\omega)-z_1\hat{\mathsf{S}}(\omega)+z_1z_2\mathsf{E}\\
\left(\hat{\mathsf{A}}(\omega)-z_1\mathsf{E}\right)\left(\hat{\mathsf{S}}(\omega)-z_2\mathsf{E}\right) &= \hat{\mathsf{S}}(\omega)\hat{\mathsf{A}}(\omega)-z_2\hat{\mathsf{A}}(\omega)-z_1\hat{\mathsf{S}}(\omega)+z_1z_2\mathsf{E}\\
\left(\hat{\mathsf{A}}(\omega)-z_1\mathsf{E}\right)\left(\hat{\mathsf{S}}(\omega)-z_2\mathsf{E}\right) &= \left(\hat{\mathsf{S}}(\omega)-z_2\mathsf{E}\right)\left(\hat{\mathsf{A}}(\omega)-z_1\mathsf{E}\right)
\end{aligned}$$

Now, by taking the inverse of both sides, it follows that the resolvents commute.

$$\begin{aligned}
\left[\left(\hat{\mathsf{A}}(\omega)-z_1\mathsf{E}\right)\left(\hat{\mathsf{S}}(\omega)-z_2\mathsf{E}\right)\right]^{-1} &= \left[\left(\hat{\mathsf{S}}(\omega)-z_2\mathsf{E}\right)\left(\hat{\mathsf{A}}(\omega)-z_1\mathsf{E}\right)\right]^{-1}\\
\left(\hat{\mathsf{S}}(\omega)-z_2\mathsf{E}\right)^{-1}\left(\hat{\mathsf{A}}(\omega)-z_1\mathsf{E}\right)^{-1} &= \left(\hat{\mathsf{A}}(\omega)-z_1\mathsf{E}\right)^{-1}\left(\hat{\mathsf{S}}(\omega)-z_2\mathsf{E}\right)^{-1}\\
\mathsf{R}_{\hat{\mathsf{S}}}(z_2,\omega)\,\mathsf{R}_{\hat{\mathsf{A}}}(z_1,\omega) &= \mathsf{R}_{\hat{\mathsf{A}}}(z_1,\omega)\,\mathsf{R}_{\hat{\mathsf{S}}}(z_2,\omega)
\end{aligned}$$

Let $\gamma_1 \in \rho\left(\hat{\mathsf{A}}(\omega)\right)$ be a closed curve that encloses only $\nu_j(\omega)$ and $\gamma_2 \in \rho\left(\hat{\mathsf{S}}(\omega)\right)$ be a closed curve that encloses only $\lambda_k(\omega)$. Let $f_1(z_1)$ and $f_2(z_2)$ be two holomorphic functions on an open set which includes the curves γ_1 and γ_2 respectively. Then

$$\begin{aligned}
&\left(-\frac{1}{2\pi i}\right)^2 \oint_{\gamma_1}\oint_{\gamma_2} f_1(z_1)f_2(z_2)\mathsf{R}_{\hat{\mathsf{S}}}(z_2,\omega)\,\mathsf{R}_{\hat{\mathsf{A}}}(z_1,\omega)\,dz_1dz_2\\
&\quad= \left(-\frac{1}{2\pi i}\right)^2 \oint_{\gamma_1}\oint_{\gamma_2} f_1(z_1)f_2(z_2)\mathsf{R}_{\hat{\mathsf{A}}}(z_1,\omega)\,\mathsf{R}_{\hat{\mathsf{S}}}(z_2,\omega)\,dz_1dz_2
\end{aligned}$$

The order of integration can be interchanged by Fubini's theorem.[5] This allows the integrals to factor.

$$\begin{aligned}
&\left(-\frac{1}{2\pi i}\oint_{\gamma_2} f_2(z_2)\mathsf{R}_{\hat{\mathsf{S}}}(z_2,\omega)\,dz_2\right)\left(-\frac{1}{2\pi i}\oint_{\gamma_1} f_1(z_1)\mathsf{R}_{\hat{\mathsf{A}}}(z_1,\omega)\,dz_1\right)\\
&\quad= \left(-\frac{1}{2\pi i}\oint_{\gamma_1} f_1(z_1)\mathsf{R}_{\hat{\mathsf{A}}}(z_1,\omega)\,dz_1\right)\left(-\frac{1}{2\pi i}\oint_{\gamma_2} f_2(z_2)\mathsf{R}_{\hat{\mathsf{S}}}(z_2,\omega)\,dz_2\right)
\end{aligned} \tag{28}$$

Recall the functional definition of the projection [17],

$$\mathsf{P}_k(\omega) = -\frac{1}{2\pi i}\oint_{\gamma_k} \mathsf{R}_{\mathsf{S}}(z,\omega)\,dz, \tag{29}$$

and of the nilpotent

$$\mathsf{N}_k(\omega) = -\frac{1}{2\pi i}\oint_{\gamma_k} (z-\lambda_k(\omega))\,\mathsf{R}_{\mathsf{S}}(z,\omega)\,dz. \tag{30}$$

where γ_k is a closed curve around $\lambda_k(\omega)$.

For $f_1 = f_2 = 1$, Eq. (28) produces $\mathsf{P}_k\mathsf{Q}_j = \mathsf{Q}_j\mathsf{P}_k$, that is Eq. (23).

---

[5] The resolvent is an analytic function on the resolvent set [8], which means that the integrals are bounded on γ_1 and γ_2.

For $f_1(z_1) = z_1 - \nu_j(\omega)$, $f_2 = 1$, Eq. (28) yields $\mathsf{P}_k\mathsf{M}_j = \mathsf{M}_j\mathsf{P}_k$, that is (24).

Similarly, the choice $f_1 = 1$ and $f_2(z_2) = z_2 - \lambda_k(\omega)$ turns Eq. (28) into Eq. (25), whereas the choice $f_1(z_1) = z_1 - \nu_j(\omega)$ and $f_2(z_2) = z_2 - \lambda_k(\omega)$ turns Eq. (28) into Eq. (26). □

Theorem 5 does not provide a constructive means by which to design shift-invariant filters. It provides a geometric constraint. For practical purposes, Thm 5 is trivially satisfied by fixing the projections of A to match those of S,

$$\hat{\mathsf{A}}(\omega) = \sum_{k=0}^{m(\omega)} \hat{a}_k(\omega)\mathsf{P}_k(\omega) + \mathsf{N}_k(\omega). \tag{31}$$

This leaves $\mathscr{O}(n)$ design parameters, the eigenvalues of $\hat{\mathsf{A}}(\omega)$ for each $\omega \in [0,1]$. More precisely, the design parameters correspond to the simple eigenvalues of $\hat{\mathsf{S}}$.[6]

However, the complications begin here. In the proof of Theorem 5, the Jordan spectral representation only holds pointwise for $\omega \in [0,1]$. In general, not much can be said about the form of the eigenvalues, projections, and nilpotents for the Jordan spectral representation of an arbitrary graph-shift operator. As was shown in Theorem 4, the entries of $\hat{\mathsf{S}}$ can be any function $L^\infty([0,1])$, and the eigenvalues, projections, and nilpotents are functions of the entries of $\hat{a}_{j,k}$, which can be drawn from a large class of functions that includes many poorly behaved functions. How the eigenvalues and invariant subspaces of an operator-valued function change as a function of some argument falls into the study of perturbation theory [17]. Most results deal in the realm of inequalities and special cases. The effect of perturbations on eigenvalues is better understood, but even for the case in which $\hat{\mathsf{S}}(\omega)$ is a continuous function of ω, there is no guarantee of unique continuous eigenfunctions. The algebraic and geometric multiplicity of eigenvalues can change with perturbations. Moreover, projection operators can blow-up as invariant subspaces collapse and reappear. For further reading, see e.g. [17]. All of this is to say that further analysis on the design of shift-invariant operators must be very deliberate and cautious.

Stronger assumptions about the form of S can lead to more well-behaved eigenvalues, projections, and nilpotents for $\hat{\mathsf{S}}$, namely holomorphicity. These results are captured in the following theorem.

Theorem 6. *Let $\mathsf{S} \in \mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)$ be a Laurent operator such that for $\varepsilon > 0$ there exist constants $c_1, c_2 > 0$ such that*

$$\|\mathsf{K}_t\|_{\mathscr{B}\left(\ell^2(\mathscr{V})\right)} \le \frac{c_1}{(1+\varepsilon)^t} \tag{32}$$

for all $t > 0$, and

$$\|\mathsf{K}_t\|_{\mathscr{B}\left(\ell^2(\mathscr{V})\right)} \le c_2(1-\varepsilon)^t \tag{33}$$

for all $t < 0$. Then, the analytic continuation of $\hat{\mathsf{S}}(\omega)$,

[6] If $\hat{\mathsf{S}}$ is degenerate, then there could be greater flexibility in the design of shift-invariant filters.

$$\hat{\hat{\mathsf{S}}}(z) = \sum_{t=0}^{\infty} z^t \mathsf{K}_t, \tag{34}$$

is a holomorphic matrix-valued function on $U = \{z \in \mathbb{C} \mid 1-\varepsilon < |z| < 1+\varepsilon\}$*. Moreover, there are n holomorphic functions* $\{\lambda_k : U \to \mathbb{C}\}_{k\in\mathscr{V}}$ *with at most algebraic singularities such that*

$$\det\left(\hat{\hat{\mathsf{S}}}(z) - \lambda_k(z)\mathsf{E}\right) = 0 \tag{35}$$

for all $k \in \mathscr{V}$ *and* $z \in U$.

Proof. Convergence will be shown element-wise in $\hat{\hat{\mathsf{S}}}$, which yields sequences $\left\{[\mathsf{K}_t]_{j,k}\right\}_{k\in\mathbb{N}}$. The Laurent expansion converges to a holomorphic function on an annulus $z \in \{z \in \mathbb{C} \mid r < |z| < R\}$ where.

$$R^{-1} = \limsup_{t\to\infty} \left|[\mathsf{K}_t]_{j,k}\right|^{1/t} \tag{36}$$

for $t > 0$, and

$$r = \limsup_{t\to-\infty} \left|[\mathsf{K}_t]_{j,k}\right|^{1/t} \tag{37}$$

for $t < 0$ for all $j,k \in \mathscr{V}$. For the stated decay rate of the kernels, this condition is satisfied on $1-\varepsilon < |z| < 1+\varepsilon$. The holomorphicity of the spectrum and nature of the singularities follows from Eq. (35), an algebraic equation for which the solutions vary analytically as a function of the elements of $\hat{\hat{\mathsf{S}}}$ (see e.g. [19]). □

Theorem 6 states that for operators with kernels that decay sufficiently fast, $\hat{\mathsf{S}} : [0,1] \to \mathscr{B}\left(\ell^2(V)\right)$ can be analytically continued on an annulus that includes the torus ($\{z \in \mathbb{C} \mid |z| = 1\}$). This guarantees holomorphicity of $\hat{\mathsf{S}}(\omega) = \hat{\hat{\mathsf{S}}}(\mathrm{e}^{2\pi i\omega})$, which is the restriction of $\hat{\hat{\mathsf{S}}}$ to the torus. That is to say that $\hat{\mathsf{S}}$ is a holomorphic function of $\omega \in [0,1]$. The significance of this result owes to the considerably stronger results on analytic perturbations of operators [17].

For analytic perturbations of finite-dimensional linear operators, the eigenvalue functions form λ-groups. These groups of eigenvalues are the multi-valued complex functions in the spectrum. For the purposes of this chapter, each λ-group along with any other λ-group it intersects in the complex plane has an associated total projection. Let $\hat{\mathsf{A}}(\omega) \in \mathscr{B}\left(\ell^2(\mathscr{V})\right)$. A projection associated with an eigenvalue $\lambda_k(\omega)$ is defined functionally for a closed curve $\Gamma_{k,\omega} \subset \mathbb{C}$ as

$$\mathsf{P}_k(\omega) = -\frac{1}{2\pi i}\oint_{\Gamma_{k,\omega}} \mathsf{R}_{\hat{\mathsf{A}}(\omega)}(z)\, dz. \tag{38}$$

For total projections, the closed curve is drawn so as to include the entire λ-group and any other intersecting λ-group. The total projection corresponds to the sum of the constituent projections, but the total projection is bounded and holomorphic on the annulus of holomorphy to include exceptional points. See Fig. 2 for further explanation of total projections. These results can be found in [17] and are reviewed here because the next theorem will require the concept of total projections. In the next section, this theory will be adapted for the practical design of filters.

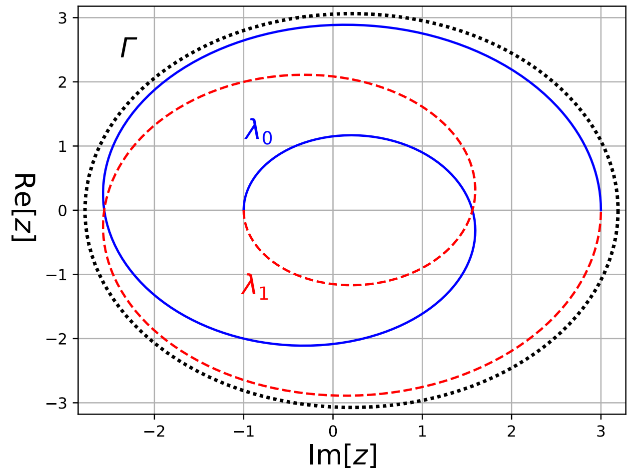

Fig. 2 Total projection of connected components of the spectrum. The spectrum of a bounded operator is the union of compact sets. Multi-valued functions (e.g. $\pm\sqrt{z}$) and odd period functions form λ-groups. The eigenvalues $\lambda_0(\omega) = e^{2\pi i\omega} + 2e^{3\pi i\omega}$ (*solid line*) and $\lambda_1(\omega) = e^{2\pi i\omega} - 2e^{3\pi i\omega}$ (*dashed line*) are plotted in the above figure. λ_0 and λ_1 together form a λ-group. This means that they cannot be partitioned by non-intersecting closed sets and require a single total projection represented by the closed curve Γ (*dotted line*). The intersections $\lambda_0(\omega) = \lambda_1(\omega)$ for $\omega \in [0,1]$ correspond to algebraic multiplicities.

4.3 Design of Linear and Shift-Invariant Filters on $\ell^2(\mathbb{Z} \times \mathscr{V})$

As in Section 3, the design of shift-invariant filters can be carried-out in the frequency domain where the action of the operator is the usual matrix-vector product. $\hat{\mathsf{A}}$ acts like a transfer function as in Eq. (14). However, in the case of shift-invariant filters, there are $\mathscr{O}(n)$ parameters because the invariant subspaces are fixed by the graph-shift operator as in Eq. (31),

$$(\mathsf{A}\mathbf{x})[t] = \left(\mathscr{F}^* \left[\left(\sum_{k=0}^{m(\omega)} \hat{a}_k(\omega)\mathsf{P}_k(\omega) + \mathsf{N}_k(\omega)\right)\hat{\mathbf{x}}(\omega)\right]\right)[t]. \tag{39}$$

As discussed above, Eq. (39) is in general very difficult to compute. This section will investigate two special cases of graph-shift operators for which concrete results can be stated: $\hat{\mathsf{S}}(\omega)$ constant and analytic.

For the case of $\hat{\mathsf{S}}(\omega)$ constant, it will help to proceed under the assumption that $\hat{\mathsf{S}}(\omega) = \hat{\mathsf{S}}$ has simple eigenvalues.

$$\hat{\mathsf{S}} = \sum_{k \in \mathscr{V}} \lambda_k \mathsf{P}_k \tag{40}$$

Extending the following results to the more general cases of semi-simple and degenerate eigenvalues is straight-froward but requires greater precision in the statement of theorems and proofs. Before proceeding with the theorem, consider the action of such a filter $\mathsf{S} = \mathscr{F}^* \hat{\mathsf{S}} \mathscr{F}$:

$$(\mathsf{S}\mathbf{x})[t] = \hat{\mathsf{S}}\mathbf{x}[t]. \tag{41}$$

S is an infinite-dimensional block diagonal operator with $\hat{\mathsf{S}}$ on the main diagonal. It has kernel function, $\mathsf{K}_0 = \hat{\mathsf{S}}$ and $\mathsf{K}_t = 0$ for $t \neq 0$.

$$\mathsf{S}\mathbf{x} = \begin{bmatrix} \ddots & & & \\ & \hat{\mathsf{S}} & & & \\ & & \hat{\mathsf{S}} & & \\ & & & \hat{\mathsf{S}} & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \mathbf{x}[-1] \\ \mathbf{x}[0] \\ \mathbf{x}[1] \\ \vdots \end{bmatrix} \tag{42}$$

This would correspond to a graph-shift operator which acts only in space, and it can be modeled as a factor graph, $\mathsf{S} = I_{\mathbb{Z}} \otimes \hat{\mathsf{S}}$.

Theorem 7. *Let $\hat{\mathsf{S}} \in \mathscr{B}\left(\ell^2(\mathscr{V})\right)$ be a constant-valued operator with a Jordan spectral representation given by Eq.* (40) *and $\mathsf{S} = \mathscr{F}^* \hat{\mathsf{S}} \mathscr{F}$. Then, for*

$$\hat{A}(\omega) = \sum_{k \in \mathscr{V}} \hat{a}_k(\omega) \mathsf{P}_k, \tag{43}$$

$\mathsf{A} = \mathscr{F}^* \hat{A} \mathscr{F}$ is shift-invariant to S, and $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z} \times \mathscr{V})\right)$ if and only if $\hat{a}_k \in L^\infty([0,1])$ for all $k \in \mathscr{V}$. Moreover, $\sigma(\mathsf{A}) = \cup_{\omega \in [0,1]} \{\hat{a}_k(\omega) \mid k \in \mathscr{V}\}$.

Proof. A is shift-invariant to S by Theorem 5.[7]

$\Leftarrow$

Let $\hat{a}_k \in L^\infty([0,1])$ for all $k \in \mathscr{V}$. It is necessary to show that A is bounded.[8]

[7] Note that $\hat{\mathsf{A}}$ could be degenerate as in Theorem 5 by making $\hat{a}(\omega) = 0$ for some $k \in \mathscr{V}$ and $\omega \in [0,1]$, and this would still be a shift-invariant operator. Also, algebraic multiplicity (i.e. $\hat{a}_j(\omega) = \hat{a}_k(\omega)$ for $j \neq k \in \mathscr{V}$) would not pose a problem as the geometric multiplicity would match that of the algebraic multiplicity.

[8] It is tacitly assumed that the projections of a bounded operator $\hat{\mathsf{S}}$ are bounded.

$$\begin{aligned}
\|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} &= \operatorname*{ess\,sup}_{\omega\in[0,1]} \left\|\hat{\mathsf{A}}(\omega)\right\|_{\mathscr{B}\left(\ell^2(\mathscr{V})\right)} \\
&= \operatorname*{ess\,sup}_{\omega\in[0,1]} \left\|\sum_{k\in\mathscr{V}} \hat{a}_k(\omega)\mathsf{P}_k\right\|_{\mathscr{B}\left(\ell^2(\mathscr{V})\right)} \\
&\le \operatorname*{ess\,sup}_{\omega\in[0,1]} \sum_{k\in\mathscr{V}} |\hat{a}_k(\omega)|\,\|\mathsf{P}_k\|_{\mathscr{B}\left(\ell^2(\mathscr{V})\right)} \\
&\le \sum_{k\in\mathscr{V}} \|\hat{a}_k(\omega)\|_{L^\infty([0,1])}\,\|\mathsf{P}_k\|_{\mathscr{B}\left(\ell^2(\mathscr{V})\right)} \\
&< \infty
\end{aligned}$$

$\Rightarrow$

Now, suppose that A is bounded and shift-invariant to S. It is necessary to show that $\|\hat{a}_k(\omega)\|_{\ell^\infty([0,1])} < \infty$ for all $k \in \mathscr{V}$, but suppose instead that it is not true for some $k' \in \mathscr{V}$. The goal is to find a $\|\tilde{\mathbf{x}}\|_{\ell^2(\mathbb{Z}\times\mathscr{V})} = 1$ for which $\|\mathsf{A}\tilde{\mathbf{x}}\|_{\ell^2(\mathbb{Z}\times\mathscr{V})} = \infty$. This can be done by choosing $\hat{\tilde{\mathbf{x}}}$ in the invariant subspace of $\mathsf{P}_{k'}$ and supported only on the set on which $\hat{a}_{k'}$ achieves its essential supremum, $\Omega^* \in [0,1]$ as shown below.

$$\begin{aligned}
\|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} &\ge \left\|\sum_{k\in\mathscr{V}} \hat{a}_k\mathsf{P}_k\hat{\tilde{\mathbf{x}}}\right\|_{L^2([0,1]\times\mathscr{V})} \\
&= \|\hat{a}_{k'}\|_{L^\infty([0,1])}\left\|\mathsf{P}_{k'}\hat{\tilde{\mathbf{x}}}\right\|_{L^2([0,1]\times\mathscr{V})}
\end{aligned}$$

Since $\hat{\tilde{\mathbf{x}}}$ was selected arbitrarily in the invariant subspace associated with $\mathsf{P}_{k'}$, let it be chosen to maximize the operator norm. Then,

$$\|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} \ge \|\hat{a}_{k'}\|_{L^\infty([0,1])}\,\|\mathsf{P}_{k'}\|_{\mathscr{B}\left(L^2([0,1]\times\mathscr{V})\right)},$$

which is not bounded. Then, it is possible to define a sequence of functions which converge to such a $\tilde{\mathbf{x}}$ which contradicts the assumption. The spectrum of A follows immediately from Theorem 3. □

Consider the operator A from Theorem 7. Following Eq. (14), its action on a signal $\mathbf{x} \in \ell^2(\mathbb{Z}\times\mathscr{V})$ takes a special form.

$$
\begin{aligned}
(\mathsf{A}\mathbf{x})[t] &= \left(\mathscr{F}^*\left[\hat{\mathsf{A}}(\omega)\hat{\mathbf{x}}(\omega)\right]\right)[t] \\
&= \int_{[0,1]} \mathrm{e}^{-2\pi i\omega t}\hat{\mathsf{A}}(\omega)\hat{\mathbf{x}}(\omega)d\omega \\
&= \int_{[0,1]} \mathrm{e}^{-2\pi i\omega t}\left(\sum_{k\in\mathscr{V}}\hat{a}_k(\omega)\mathsf{P}_k\right)\left(\sum_{s\in\mathbb{Z}}\mathrm{e}^{2\pi i\omega s}\mathbf{x}[s]\right)d\omega \\
&= \sum_{k\in\mathscr{V}}\sum_{s\in\mathbb{Z}}\left(\int_{[0,1]}\mathrm{e}^{2\pi i\omega(s-t)}\hat{a}_k(\omega)d\omega\right)\mathsf{P}_k\mathbf{x}[s] \\
&= \sum_{k\in\mathscr{V}}\left(\sum_{s\in\mathscr{V}}a_k[s-t]\mathsf{P}_k\mathbf{x}[s]\right) \\
&= \sum_{k\in\mathscr{V}}\left(a_k * \mathsf{P}_k\mathbf{x}\right)[t]
\end{aligned}
$$

Here, $a_k[t] = (\mathscr{F}^*\hat{a}_k)[t]$. If $\hat{a}_k \in L^2([0,1]) \cap L^\infty([0,1])$, then $a_k \in \ell^2(\mathbb{Z})$.[9] The action of A in the time-domain first projects the time-indexed signal $\mathbf{x}[t] \in \ell^2(\mathscr{V})$ onto subspaces defined by the projection operators, P_k and then convolves the signal with a function $a_k \in \ell^2(\mathbb{Z})$. Here, $\hat{a}_k$ acts as transfer function on the projected signal.

Now, shift-invariance with respect to holomorphic operators is considered.

Theorem 8. *Let $\mathsf{S} \in \mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)$ satisfy the conditions of Theorem 6, and let $\hat{\mathsf{S}} : [0,1] \to \ell^2(\mathscr{V})$ have total projection operators $\{\mathsf{P}_k(\omega) \mid k = 0,\ldots,m(\omega)\}$ where $0 < m(\omega) \leq n = |\mathscr{V}|$. Then, for*

$$
\hat{\mathsf{A}}(\omega) = \sum_{k=0}^{m(\omega)} \hat{a}_k(\omega)\mathsf{P}_k(\omega), \tag{44}
$$

$\mathsf{A} = \mathscr{F}^*\hat{\mathsf{A}}\mathscr{F}$ is shift-invariant to S and $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)$ if and only if $\hat{a}_k \in L^\infty([0,1])$ for all $k \in \mathscr{V}$. Moreover, $\sigma(\mathsf{A}) = \cup_{\omega\in[0,1]}\{\hat{a}_k(\omega) \mid k = 0,\ldots,m(\omega)\}$.

Proof. The proof follows that of Theorem 7 by the boundedness of the total projections [17]. □

How realistic is it to expect a graph-shift operator which admits an analytic continuation? In almost all practical applications, it would in fact be the case. When modeling the weighted edges of an extended graph from a real-world process, it would be realistic to assume that the impact of events would decay to a negligible effect within finite time or that the conditional probabilities across time would be negligible for large enough time. This would satisfy the conditions of Theorem 6.

9 Since $\mathscr{F} : \ell^1(\mathbb{Z}) \to L^\infty([0,1])$ is not surjective, $\mathscr{F}^*\hat{a}_k$ may not exist for some $\hat{a}_k \in L^\infty([0,1])$. However, $\mathscr{F} : \ell^2(\mathbb{Z}) \to L^2([0,1])$ is bijective.

4.4 Example

We aim to implement a bandpass filter on a signal $\mathbf{x} \in \ell^2(\mathbb{Z} \times \mathscr{V})$ using the framework of Section 4.3. In Sandryhaila and Moura [35], bandpass filtering on graph signals is defined according to the eigenvectors of the weighted adjacency matrix W and ordered by the so called graph total variation. A low-pass filter allows the projections of the signal onto the subspaces associated with small total variation eigenvectors to pass, and a high-pass filter the projections associated with high total variation eigenvectors. As in Section 3.4, a bandpass filter can attenuate unwanted frequencies of the time-series graph signal, but now, a bandpass filter can also attenuate subspaces of the graph signal.

Consider the example of Fig. 1, and let the kernel function be defined as follows:

$$\mathsf{K}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathsf{K}_1 = \begin{bmatrix} -\frac{2}{5} & 0 \\ 0 & -\frac{2}{5} \end{bmatrix} \quad \mathsf{K}_2 = \begin{bmatrix} 0 & 0 \\ \frac{4}{5} & 0 \end{bmatrix} \quad \mathsf{K}_3 = \begin{bmatrix} 0 & \frac{2}{5} \\ 0 & 0 \end{bmatrix} \tag{45}$$

and $\mathsf{K}_t = 0$ otherwise, i.e. we assign weights to the edges of Fig. 1. By Theorem 3, this results in a spectral representation of the graph-shift operator,

$$\hat{\mathsf{S}}(\omega) = \begin{bmatrix} -\frac{2}{5}\mathrm{e}^{2\pi i\omega} & 1+\frac{2}{5}\mathrm{e}^{6\pi i\omega} \\ 1+\frac{4}{5}\mathrm{e}^{4\pi i\omega} & -\frac{2}{5}\mathrm{e}^{2\pi i\omega} \end{bmatrix}, \tag{46}$$

which is holomorphic on $\omega \in [0,1]$ by Theorem 6. The spectral operator has eigenvalues,

$$\lambda_{\pm}(\omega) = -\frac{2}{5}\mathrm{e}^{2\pi i\omega} \pm \sqrt{\left(1+\frac{4}{5}\mathrm{e}^{4\pi i\omega}\right)\left(1+\frac{2}{5}\mathrm{e}^{6\pi i\omega}\right)}, \tag{47}$$

projection operators,

$$\mathsf{P}_{\pm}(\omega) = \pm\frac{1}{2}\begin{bmatrix} 1 & \pm\sqrt{\frac{5+2\mathrm{e}^{6\pi i\omega}}{5+4\mathrm{e}^{4\pi i\omega}}} \\ \pm\sqrt{\frac{5+4\mathrm{e}^{4\pi i\omega}}{5+2\mathrm{e}^{6\pi i\omega}}} & 1 \end{bmatrix}, \tag{48}$$

and nilpotents $\mathsf{N}_{\pm}(\omega) = 0$ respectively.

Now consider the bandpass filter. Suppose that we are to pass the frequency content on $\Omega \in [0,1]$, and that the graph signal component associated with $\mathsf{P}_{-}(\omega)$ is to be attenuated. Then, an ideal bandpass filter $\mathsf{A} = \mathscr{F}^*\hat{\mathsf{A}}\mathscr{F}$ is defined by

$$\hat{\mathsf{A}}(\omega) = \begin{cases} \mathsf{P}_{+}(\omega) & \omega \in \Omega \\ 0 & o.w. \end{cases}. \tag{49}$$

This would correspond to choosing $\hat{a}_{+}(\omega) = \chi_{\Omega}(\omega)$ and $\hat{a}_{-}(\omega) = 0$, which are bounded functions on $[0,1]$. Therefore, $\mathsf{A} \in \mathscr{B}\left(\ell^2(\mathbb{Z} \times \mathscr{V})\right)$ by Theorem 8. The action of A would be

$$(\mathsf{A}\mathbf{x})[t] = \int_{\Omega} \mathrm{e}^{-2\pi i \omega t} \mathsf{P}_{+}(\omega)\hat{\mathbf{x}}(\omega) d\omega. \tag{50}$$

In the limit as $d\omega(\Omega) \to 0$ so that $\Omega = \{\omega_0\}$, this would implement an ideal band-pass,[10]

$$(\mathsf{A}\mathbf{x})[t] = \mathrm{e}^{-2\pi i \omega_0 t} \mathsf{P}_{+}(\omega_0)\hat{\mathbf{x}}(\omega_0). \tag{51}$$

5 Functions of Graph-Shift Operators on $\ell^2(\mathbb{Z} \times \mathscr{V})$

Shift-invariant filters enable the incorporation of prior information in the form of graph-shift operators into the design process. However, linear scaling of the design parameters does not take advantage of sparsity, nor does it scale to large graphs with $|\mathscr{V}| \gg 0$. This begs for an alternative filter design method with greater control on the design complexity.

This section will present the theory for designing filters which are themselves functions of graph-shift operators. In the context of graph signal processing, this concept emerges as defining filters which are polynomials of a given graph-shift operator. Sandryhaila and Moura [33] observed that in finite-dimensions, all shift-invariant filters can be realized as finite degree polynomials of the graph-shift operator. The concept of defining filters as polynomials of a given graph-shift operator arose also in Shuman, et al. [38] and again in Defferrard, et al. [6]. Defining filters as polynomials of graph-shift operators brings with it a strong physical interpretability as graph operators act only on a neighborhood of each node. Thus, each polynomial order connotes a different physical scale of the action of an operator. Moreover, choosing only the coefficients of polynomials gives the designer much greater control on the complexity of the design problem and allows much of the computationally intensive portions of filtering to be computed offline.

5.1 Functional Calculus

In spectral theory, polynomials of finite-dimensional matrices is only a subset of a more comprehensive theory of functional calculus. The intuition is straight-forward. Power series allow one to arbitrarily approximate scalar functions with infinite polynomials, and polynomials of matrices act on the eigenvalues. In the case of matrices, the Cayley-Hamilton theorem states that for a matrix $\mathsf{A} \in \mathbb{C}^{n \times n}$, all matrices A^k for $k \geq n$ are in the span of $\{\mathsf{A}^k \mid k = 0, \ldots, n-1\}$. Thus, it is unnecessary to consider polynomials of degree greater than $n-1$ [12]. However, infinite-dimensional operators are not constrained by this result, and admit full power series. Polynomials are dense in the space of continuous functions, so arbitrary degree polynomials of

[10] Again, this converges only in the distribution sense as it amounts to $\hat{a}_{+}(\omega) \to \delta(\omega - \omega_0)$, a non-measurable function.

operators can potentially yield a much larger class of functions. Functional calculus provides the theory by which to define operators that are functions (to include polynomials) of other operators. In general, there is the holomorphic functional calculus. For a proof, see e.g. [8].

Theorem 9 (Holomorphic functional calculus). *Let* $\mathsf{S} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z}\times\mathscr{V}\right)\right)$*,* $U \subset \mathbb{C}$ *be an open set such that* $\sigma(\mathsf{S}) \subset U$*,* $\phi : U \to \mathbb{C}$ *be holomorphic, and* $\Gamma \subset U$ *be a closed curve enclosing* $\sigma(\mathsf{S})$*. Then, there exists a* $\phi(\mathsf{S}) \in \mathscr{B}\left(\ell^2\left(\mathbb{Z}\times\mathscr{V}\right)\right)$ *such that*

$$\phi(\mathsf{S}) := -\frac{1}{2\pi i}\int_{\Gamma}\phi(z)\mathsf{R}_{\mathsf{S}}(z)\,dz. \tag{52}$$

Moreover, $\sigma(\phi(\mathsf{S})) = \phi(\sigma(\mathsf{S}))$*,* $\phi \mapsto \phi(\mathsf{S})$ *is a continous map from* $\sup_{\gamma\in\Gamma}|\phi(\gamma)|$ *to* $\|\cdot\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)}$*, and if* $\psi : \mathbb{C}\to\mathbb{C}$ *is holomorphic on* U*, then* $\phi(\mathsf{S})\psi(\mathsf{S}) = (\phi\circ\psi)(\mathsf{S})$*.*

Hidden in the statement of the theorem is that the open set can be the union of finitely many disjoint open sets so that each connected component of the spectrum could have its own open set $U = \cup_{k=0}^{m} U_k$. Moreover, $\phi : U \to \mathbb{C}$ only needs to be holomorphic on U. That is to say that it must admit a power series representation at all points $z \in U$, but the power series representations need not correspond on disjoint open sets. The restriction of ϕ to U_0 could be one holomorphic function and another for the restriction of ϕ to U_1. Moreover, it need not have an analytic extension on $\mathbb{C}\setminus(U_0\cup U_1)$. See Fig. 3 for further explanation.

5.2 Theoretical Results on Filters that are Functions of Graph-Shift Operators

Ultimately, the goal of this chapter is to design linear and time-invariant filters for time-varying graph signals. In Section 4, it was shown that shift-invariance incorporates spatial information in the form of a graph operator into the design of filters. In this section, the goal is to maintain that information through shift-invariance to a given graph-shift operator.

Theorem 10. *Let* $\mathsf{S} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z}\times\mathscr{V}\right)\right)$ *be Laurent,* $U \subset \mathbb{C}$ *be an open set such that* $\sigma(\mathsf{S}) \subset U$*, and* $\phi : U \to \mathbb{C}$ *be a holomorphic function. If* $\mathsf{A} = \phi(\mathsf{S})$*, then* $\mathsf{A} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z}\times\mathscr{V}\right)\right)$ *is shift-invariant to* S *where* $\phi(\mathsf{S})$ *is defined according to Theorem 9.*

Proof. We begin with shift-invariance. It will help to first prove that S commutes with $\mathsf{R}_{\mathsf{S}}(z)$ for all $z \in \rho(\mathsf{S})$.

$$\begin{aligned}
\mathsf{S}(\mathsf{S}-z\mathsf{E}) &= (\mathsf{S}-z\mathsf{E})\,\mathsf{S} \\
\mathsf{R}_{\mathsf{S}}(z)\,\mathsf{S}(\mathsf{S}-z\mathsf{E})\,\mathsf{R}_{\mathsf{S}}(z) &= \mathsf{R}_{\mathsf{S}}(z)\,(\mathsf{S}-z\mathsf{E})\,\mathsf{S}\mathsf{R}_{\mathsf{S}}(z) \\
\mathsf{R}_{\mathsf{S}}(z)\,\mathsf{S} &= \mathsf{S}\mathsf{R}_{\mathsf{S}}(z)
\end{aligned}$$

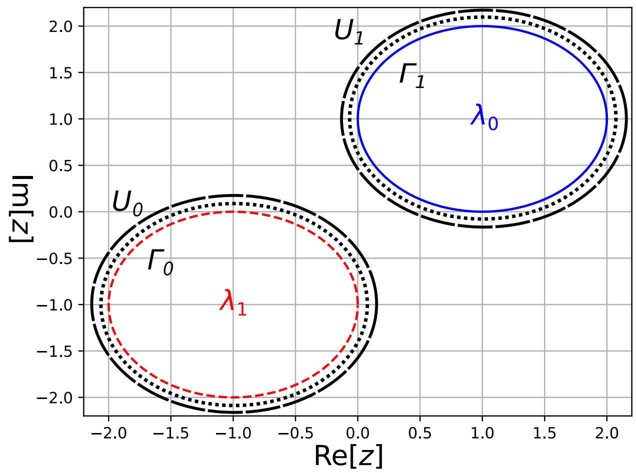

Fig. 3 Example domain for holomorphic functional calculus. Here, the spectrum comprises two disjoint sets $\{\lambda_0(\omega) \in \mathbb{C} : \omega \in [0,1]\}$ (*solid line*) and $\{\lambda_1(\omega) \in \mathbb{C} : \omega \in [0,1]\}$ (*dashed line*). A holomorphic function $\phi : U \to \mathbb{C}$ must be holomorphic on an open set $U \subset \mathbb{C}$, but that set can be the union of open sets, i.e. $U = U_0 \cup U_1$ (*long dashed line*) in the figure. Moreover, ϕ only needs to be holomorphic on U, which means that the restriction of ϕ to U_0 and the restriction of ϕ to U_1 can be different holomorphic functions, and they do not have to be holomorphic on $\mathbb{C} \setminus U$. The integration is done on an arbitrary curve $\Gamma \subset U$ that can be the union of curves $\Gamma = \Gamma_0 \cup \Gamma_1$ (*dotted line*).

Now, consider shift-invariance

$$\begin{aligned}
\mathsf{AS} &= \phi\left(\mathsf{S}\right)\mathsf{S} \\
&= \left(\oint_{\Gamma} \phi(z)\mathsf{R}_{\mathsf{S}}\left(z\right)dz\right)\mathsf{S} \\
&= \oint_{\Gamma} \phi(z)\left(\mathsf{R}_{\mathsf{S}}\left(z\right)\mathsf{S}\right)dz \\
&= \oint_{\Gamma} \phi(z)\left(\mathsf{S}\mathsf{R}_{\mathsf{S}}\left(z\right)\right)dz \\
&= \mathsf{S}\left(\oint_{\Gamma} \phi(z)\mathsf{R}_{\mathsf{S}}\left(z\right)dz\right) \\
&= \mathsf{S}\phi\left(\mathsf{S}\right) \\
&= \mathsf{SA}
\end{aligned}$$

Lastly, it is necessary to show that $\|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} < \infty$.

$$\begin{aligned}
\|\phi(\mathsf{S})\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} &= \left\| \oint_\Gamma \phi(z) \mathsf{R}_\mathsf{S}(z)\, dz \right\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} \\
&\le \sup_{z\in\Gamma} |\phi(z)| \left\| \oint_\Gamma \mathsf{R}_\mathsf{S}(z)\, dz \right\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} \\
&\le M \sup_{z\in\Gamma} |\phi(z)| \\
&< \infty
\end{aligned}$$

ϕ is bounded by the maximum modulus theorem [2], and the third equality comes from Prop. 4 of Theorem 1. □

This shows that filters defined as functions of graph-shift operators are indeed shift-invariant; however, Theorem 10 is not constructive. The following theorem provides that.

Theorem 11. *Let $\mathsf{S} \in \mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)$ be Laurent, $U \subset \mathbb{C}$ be an open set such that $\sigma(\mathsf{S}) \subset U$, and $\phi : U \to \mathbb{C}$ be a holomorphic function. Further, let $\hat{\mathsf{S}}$ have a Jordan spectral representation given pointwise by*

$$\hat{\mathsf{S}}(\omega) = \sum_{k=0}^{m(\omega)} \lambda_k(\omega) \mathsf{P}_k(\omega) + \mathsf{N}_k(\omega) \tag{53}$$

for $\omega \in [0,1]$ and $0 < m(\omega) \le n$. Then, for $\mathbf{x} \in \ell^2(\mathbb{Z}\times\mathscr{V})$ and $\mathsf{A} = \phi(\mathsf{S})$,

$$(\mathsf{A}\mathbf{x})[t] = \left(\mathscr{F}^* \left\{ \sum_{k=0}^{m(\omega)} \left[(\phi\circ\lambda_k)(\omega) \mathsf{P}_k(\omega) + \left(\phi'\circ\lambda_k\right)(\omega) \mathsf{N}_k(\omega) \right] \hat{\mathbf{x}}(\omega) \right\} \right)[t]. \tag{54}$$

Moreover, $\sigma(\mathsf{A}) = \cup_{\omega\in[0,1]} \{(\phi\circ\lambda_k)(\omega) \mid k = 0,\ldots,m(\omega)\}$.

Proof. The Jordan spectral representation of the resolvent for Eq. (53) has the following form:

$$\mathsf{R}_{\hat{\mathsf{S}}}(z,\omega) = \sum_{k=1}^{m(\omega)} (z-\lambda_k(\omega))^{-1} \mathsf{P}_k(\omega) + (z-\lambda_k(\omega))^{-2} \mathsf{N}_k(\omega) \tag{55}$$

for $z \in \rho\left(\hat{\mathsf{S}}(\omega)\right)$ and $\omega \in \mathbb{C}$ [17]. It is helpful to first establish the following identity.

$$\begin{aligned}
(\mathscr{F}\phi(\mathsf{S})\mathbf{x})(\omega) &= \left(\mathscr{F} \left[-\frac{1}{2\pi i} \oint_\Gamma \phi(z) \mathsf{R}_\mathsf{S}(z)\, dz \right] \mathbf{x} \right)(\omega) \\
&= \left(-\frac{1}{2\pi i} \oint_\Gamma \phi(z) \left(\mathscr{F} (\mathsf{S} - z\mathsf{E})^{-1} \mathbf{x} \right)(\omega) dz \right) \\
&= \left(-\frac{1}{2\pi i} \oint_\Gamma \phi(z) \left[\left(\hat{\mathsf{S}}(\omega) - z\mathsf{E}\right)^{-1} \hat{\mathbf{x}}(\omega) \right] dz \right) \\
&= \left[\phi\left(\hat{\mathsf{S}}\right) \right](\omega) \hat{\mathbf{x}}(\omega)
\end{aligned}$$

The implication here is that $\phi(\mathsf{S}) = \mathscr{F}^*\phi(\hat{\mathsf{S}})\mathscr{F}$. Then, this along with the following can be used with Eq. (55) to derive the desired result.

$$\begin{aligned}
\phi\left(\hat{\mathsf{S}}\right) &= \frac{1}{2\pi i}\oint_{\Gamma}\phi(z)\mathsf{R}_{\hat{\mathsf{S}}}\left(z,\omega\right)dz \\
&= \frac{1}{2\pi i}\oint_{\Gamma}\phi(z)\left(\sum_{k=1}^{m(\omega)}\left(z-\lambda_k(\omega)\right)^{-1}\mathsf{P}_k(\omega) + \left(z-\lambda_k(\omega)\right)^{-2}\mathsf{N}_k(\omega)\right)dz \\
&= \sum_{k=1}^{m(\omega)}\frac{1}{2\pi i}\oint_{\Gamma}\left[\frac{\phi(z)}{z-\lambda_k(\omega)}\mathsf{P}_k(\omega) + \frac{\phi(z)}{\left(z-\lambda_k(\omega)\right)^2}\mathsf{N}_k(\omega)\right]dz \\
&= \sum_{k=0}^{m(\omega)}\left(\phi\circ\lambda_k\right)(\omega)\mathsf{P}_k(\omega) + \left(\phi'\circ\lambda_k\right)(\omega)\mathsf{N}_k(\omega)
\end{aligned}$$

The spectrum identity follows from Theorem 9, and it is known as the spectral mapping theorem [8]. □

5.3 Design of Filters that are Functions of Graph-Shift Operators

Although constructive, actually computing $\phi(\mathsf{S})$ requires explicit calculation of the spectrum. As discussed in Section 4.2, explicit calculation of the spectrum can be prohibitively difficult in all but special cases. Therefore, this section will proceed as in Section 4.3 with $\hat{\mathsf{S}}(\omega)$ constant.

Theorem 12. *Let $\hat{\mathsf{S}}(\omega) = \hat{\mathsf{S}} \in \mathscr{B}\left(\ell^2\left(\mathscr{V}\right)\right)$ be a constant-valued operator with a Jordan spectral representation given by*

$$\hat{\mathsf{S}}(\omega) = \hat{\mathsf{S}} = \sum_{k\in\mathscr{V}}\lambda_k\mathsf{P}_k, \tag{56}$$

and $\mathsf{S} = \mathscr{F}^*\hat{\mathsf{S}}\mathscr{F}$. Further, let $U \subset \mathbb{C}$ be an open set such that $\sigma\left(\mathsf{S}\right) \subset U$, $\phi : U \to \mathbb{C}$ be a holomorphic function, $\mathsf{A} = \phi(\mathsf{S})$. Then, for $\mathbf{x} \in \ell^2\left(\mathbb{Z}\times\mathscr{V}\right)$,

$$\left(\mathsf{A}\mathbf{x}\right)[t] = \phi\left(\hat{\mathsf{S}}\right)\mathbf{x}[t] = \sum_{k\in\mathscr{V}}\phi(\lambda_k)\mathsf{P}_k\mathbf{x}[t], \tag{57}$$

and $\sigma\left(\mathsf{A}\right) = \left\{\phi(\lambda_k)\right\}_{k\in\mathscr{V}}$.

Proof. This is a straightforward application of Theorem 11.

Theorem 12 indicates that $\phi(\mathsf{S})$ acts independent of time for a graph-shift operator S that acts independent of time as can be seen from the matrix representation.

$$\phi(\mathsf{S})\mathbf{x} = \begin{bmatrix} \ddots & & & & \\ & \phi\left(\hat{\mathsf{S}}\right) & & & \\ & & \phi\left(\hat{\mathsf{S}}\right) & & \\ & & & \phi\left(\hat{\mathsf{S}}\right) & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \mathbf{x}[-1] \\ \mathbf{x}[0] \\ \mathbf{x}[1] \\ \vdots \end{bmatrix} \tag{58}$$

The design of filters through functions of graph-shift operators admits only $\mathscr{O}(1)$ parameter, the holomorphic function $\phi : U \to \mathbb{C}$. Such operations on graph-shift operators also induce a physical meaning as actions on local neighborhoods. Graph-shifts induce local neighborhoods and pathways on a graph, and higher degree polynomials induce larger neighborhoods and pathways of correspondence between nodes on a graph. The class of holomorphic function allows the realization of fully local to fully connected operations on the graph to influence the filter.

5.4 Example

Again, we would like to implement a bandpass filter. However, in the case of function calculus, the design parameter $\phi : U \to \mathbb{C}$ is a function on the spectrum, and it cannot act explicitly on the frequency domain as in the filters of Sections 3.4 and 4.4. Therefore, more care is required to implement a bandpass filter according to Section 5.3.

Consider again the example of Fig. 1 with kernel function given in Section 4.4. The first step in designing a filter according to Theorem 11 will be defining an open set $U \subset \mathbb{C}$ such that $\sigma(\mathsf{S}) \subset U$. From Section 4.4, $\sigma(\mathsf{S}) = \cup_{\omega\in[0,1]} \{\lambda_+(\omega), \lambda_-(\omega)\}$. As seen in Fig. 4, $\lambda_+(\omega)$ and $\lambda_-(\omega)$ are disjoint on $\mathbb{C}$. This allows us to define $U = U_+ \cup U_-$ to satisfy $U_+ \cap U_- = \emptyset$ and $\cup_{\omega\in[0,1]}\lambda_+(\omega) \subset U_+$ and $\cup_{\omega\in[0,1]}\lambda_-(\omega) \subset U_-$.

Let U be defined as in Fig. 4 and suppose that as in Section 4.4, the goal is to attenuate the signal in the subspace of P_-. This can be done by letting, $\phi|_{(U_+)^c} = 0$, a holomorphic function on U_-. Now, consider a complex circular Gaussian,

$$\nu(z;\mu,\sigma) = \frac{1}{2\pi\sigma}\exp\left(-\frac{|z-\mu|^2}{2\sigma^2}\right), \tag{59}$$

with $\mu \in \mathbb{C}$ and $\sigma > 0 \in \mathbb{R}$ [30]. This also defines a holomorphic function, so let $\phi|_{U_+} = \nu(z;\mu,\sigma)$, and let $\mathsf{A} = \phi(\mathsf{S})$. Then, $\mathsf{A} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z}\times\mathscr{V}\right)\right)$ by Theorem 11 and has the following action

$$\left(\mathsf{A}\mathbf{x}\right)[t] = \int_0^1 \mathrm{e}^{-2\pi i\omega t}\nu(\lambda_+(\omega);\mu,\sigma)\mathsf{P}_+(\omega)\hat{\mathbf{x}}(\omega)d\omega. \tag{60}$$

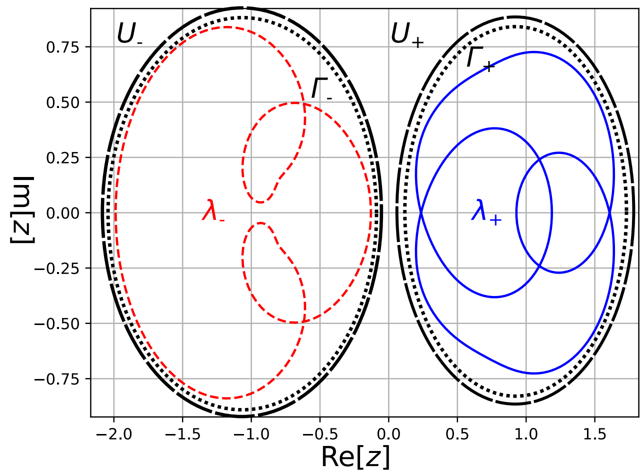

Fig. 4 Spectrum of example S of Fig. 1. The spectrum comprises two disjoint sets, $\{\lambda_+(\omega) \in \mathbb{C} : \omega \in [0,1]\}$ (*solid line*) and $\{\lambda_-(\omega) \in \mathbb{C} : \omega \in [0,1]\}$ (*dashed line*). As a result, we can define an open set $U = U_- \cup U_+$ (*long dashed line*) and closed contour $\Gamma = \Gamma_- \cup \Gamma_+$ (*dotted line*). Any holomorphic function $\phi : U \to \mathbb{C}$ need only be holomorphic on U.

In the limit, for $\mu = \lambda_+(\omega_0)$ with $\omega_0 \in [0,1]$ and $\sigma \to 0$, A has the following action[11]

$$(\mathsf{A}\mathbf{x})[t] = \mathrm{e}^{-2\pi i \omega_0 t} \mathsf{P}_+(\omega_0)\hat{\mathbf{x}}(\omega_0). \tag{61}$$

In principle, it is observed that near ideal bandpass is possible using functional calculus.

6 Special Theory for Self-Adjoint Filters

As graphs can be either directed or undirected, methods that work for either enjoy an advantage in terms of universality; however, the analysis of self-adjoint matrices admits significantly stronger results and has led much of the work on graph signal processing to be done on undirected graphs. Thus far, the proposed approach works whether the graph is directed or undirected (or equivalently the graph-shift operator is self-adjoint). This section will briefly discuss the implications of self-adjoint graph-shift operators for Sections 4 and 5.

The key advantage to analysis of self-adjoint graph-shift operators is the existence of an orthonormal basis from an eigendecomposition. In Section 4.2, this will

[11] This is not strictly admissible, but holds only in the distribution sense, since ϕ is no longer a measurable holomorphic function.

yield semi-simple real-valued eigenvalues. In Section 5.2, this will yield a larger class of functions, namely all Borel measurable functions.

It is a common result in matrix analysis that a finite-dimensional self-adjoint operator can be diagonalized by a unitary operator [12]. It then follows that the Jordan spectral representation of a self-adjoint graph-shift operator $\hat{\mathsf{S}} \in \mathscr{B}\left(\ell^2\left(\mathscr{V}\right)\right)$ can be written

$$\hat{\mathsf{S}} = \sum_{k \in \mathscr{V}} \lambda_k \mathbf{u}_k \mathbf{u}_k^* \tag{62}$$

where $\lambda_k \in \mathbb{R}$ and $\|\mathbf{u}_k\|_{\ell^2(\mathscr{V})} = 1$ for all $k \in \mathscr{V}$.[12] Additionally, the eigenvalues and operator norm of a self-adjoint operator $\mathsf{A} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z} \times \mathscr{V}\right)\right)$ are related as follows:

$$\|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} = \max_{\lambda \in \sigma(\mathsf{A})} |\lambda|. \tag{63}$$

The maximum of the spectrum is known as the spectral radius, and this follows from Gelfand's theorem [5]. This means that the norm for a shift-invariant filter $\mathsf{A} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z} \times \mathscr{V}\right)\right)$ designed according to Section 4.3 for a graph-shift $\mathsf{S} \in \mathscr{B}\left(\ell^2\left(\mathbb{Z} \times \mathscr{V}\right)\right)$ is

$$\|\mathsf{A}\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} = \max_{\omega \in [0,1], k \in \mathscr{V}} |\hat{a}_k(\omega)|. \tag{64}$$

In Shuman et al. [38], the authors propose that the orthogonal projections defined by the Laplacian substitute as the Fourier basis in a graph Fourier transform. The authors make use of the fact that one can find eigenvectors of the Laplacian L that form an orthonormal basis in $\ell^2\left(\mathscr{V}\right)$. Moreover, the basis vectors correspond to real-valued eigenvalues that can be ordered, from which the basis vectors can in turn be ordered. These results underlie the intuition behind the graph Fourier transform.

Additionally, for self-adjoint operators, there is the Borel functional calculus, which includes an even larger set of functions than the holomorphic functional calculus. For more detail on the Borel functional calculus, see e.g. [39]. The Borel functional calculus admits any Borel measurable function for self-adjoint operators. In addition to allowing a larger set of functions, the Borel functional calculus provides a control on the operator norm of the resultant operator. For a Borel measurable function ϕ,

$$\|\phi(\mathsf{A})\|_{\mathscr{B}\left(\ell^2(\mathbb{Z}\times\mathscr{V})\right)} = \max_{\lambda \in \phi(\sigma(\mathsf{A}))} |\lambda|. \tag{65}$$

[12] Eq. (62) is no longer strictly unique as it can include repeated eigenvalues and requires the choice of an appropriate basis for any geometric multiplicity.

7 Conclusion

This chapter addressed the filtering of time-varying graph signals for time-invariant extended graphs. The theoretical framework proposed yields three families of design methods for time-invariant and shift-invariant filters. Bandpass filtering was used as a design example. Future work will explore the implications of finite sampling in the time domain and pursue applications in statistical inference for social networks and brain imaging.

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