CODISTRIBUTIONS WITH SINGULARITIES: Punctual and Local Study

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Abstract

In this paper we construct a theory of codistributions with singularities that is the dual of the theory of distributions with singularities. We present the Freeman's construction, our punctual results and a new proof of a Freeman's theorem about integrability and normal form of finitely-generated module of germs of smooth 1-forms.

1 Introduction

Let M be a $\mathcal{C}^r$ finite-dimensional, connected and paracompact manifold ($r = \infty$ or ω, by the case); let $\mathcal{F}(M)$ denote the ring of $\mathcal{C}^r$ real-valued functions defined on M, let $V^r(M)$ be the $\mathcal{F}(M)$-module of $\mathcal{C}^r$ vector fields and $\Lambda^k(M)$ be the $\mathcal{F}(M)$-module of $\mathcal{C}^r$ k-forms on M.

By a *Pfaffian form* we shall mean a $\mathcal{C}^r$ 1-form. We call $\mathcal{C}^r$*-(Pfaffian) differential system* on M a $\mathcal{F}(M)$-module of $\mathcal{C}^r$ 1-forms. We shall denote it by $\mathcal{P}$. We call *codistribution* on M, the mapping: $P : x \in M \longrightarrow P(x) \subset T_x^*M$ where $P(x)$ is a vector subspace of the cotangent space to M at x. The *dimension* (or *rank*) of the codistribution is $\dim P(x)$ (it is punctually defined).

Let S be a set of $\mathcal{C}^r$ 1-forms everywhere defined. The codistribution generated by the set S is:

$$P(x) = span_{\mathbf{R}}\{\omega|_x,\ \omega \in S\}\ \forall\ x \in M.$$

We call C^r*-codistribution* on M, a codistribution P generated by a set S of C^r 1-forms. The codistribution P is called *integrable in Cartan sense* at $x_0 \in M$ if there exists a submanifold $\mathcal{N}_{\varepsilon.x_0} \overset{i}{\hookrightarrow} M$ (i being the canonical inclusion) passing through x_0, such that:

$$T_x\mathcal{N}_{\varepsilon.x_0} = (P(x))^{\perp} \ , \ for\ all\ x \in \mathcal{N}_{\varepsilon.x_0}$$

(more precisely, we have: $i^*_{,x}(P(x)) = 0$ and $\dim P(x) + \dim \mathcal{N}_{\varepsilon.x_0} = \dim M$ for all $x \in \mathcal{N}_{\varepsilon.x_0}$, where $i^*_{,x}$ is the standard pull-back of i in x). $\mathcal{N}_{\varepsilon.x_0}$ is called an *integral manifold* of the codistribution and we say also that P is *punctually integrable* at x_0. From definition it follows directly that P is also punctually integrable at every $q \in \mathcal{N}_{\varepsilon.x_0}$.

The codistribution is called *locally integrable* if for each point in M there is an integral manifold of the codistribution (namely if it is punctually integrable at every point of M).

The differential system $\mathcal{P}$ is called *integrable in Pfaff sense* if there exists a finite set of generators of exact forms (i.e. if there exist $f_i : M \to \mathbf{R}$, $1 \leq i \leq p$, such that: $\mathcal{P} = span_{\mathcal{F}(M)}\{df_1, \ldots, df_p\}$).

Let us consider the codistribution P and a point $x_0 \in M$. If there exists a neighborhood of x_0 where the codistribution has constant dimension then the point x_0 is called an *ordinary point* (or a *regular point*), otherwise it is called a *singular point*. If the codistribution has singular points then we say that it is a *codistribution with singularities.*

Our goal is to find criteria of punctual and local integrability of a codistribution generated by a $\mathcal{F}(M)$-module of C^r 1-forms (possibly having singularities) and a complete characterization of a special class of finitely-generated differential systems. This paper actually aims to etablish a dual version of a previous study [Ba92]

In §2 we give a few examples about both types of integrability.

In §3 we present the Freeman's construction and our split of codistribution. We define also the concept of involutivity of a differential system showing the connection with the involutivity of some module of smooth vector fields.

In §4 we prove our results about punctual integrability for smooth and analytic codistributions.

In §5 we give a new proof of the normal form theorem of a finitely-generated, involutive, differential system. The idea of this proof can be used to find a normal form of systems of k-forms.

Since our study is punctually or locally we point out that here the integral manifolds are regular embedding submanifolds.

2 Preliminary definitions and examples

Let $\mathcal{P}$ be a C^r differential system and P denote the generated codistribution. The following two examples prove the independence between the two types of integrability:

EXAMPLE 2.1 *$\mathcal{P}$ is integrable in Pfaff sense, but P is not integrable in Cartan sense.*

Let $f_1 = x + y$ and $f_2 = \sin x + y$, and $\mathcal{P} = span_{\mathcal{F}(M)}\{df_1, df_2\}$, $M = \mathbf{R}^2$. To find integral manifold we have the system:

$$\left\{ \begin{array}{l} \omega_1 = df_1 = 0 \\ \omega_2 = df_2 = 0 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{ccc} x+y & = & C_1 \\ \sin x + y & = & C_2 \end{array} \right.$$

where $C_1, C_2 \in \mathbf{R}$ are real constants. Since the system has a unique solution: $x = x(C_1, C_2)$ & $y = y(C_1, C_2)$ for every pair (C_1, C_2) it follows that at the origin the codistribution is not punctually integrable. $\diamond$

EXAMPLE 2.2 (**[Fr84]**) *P is integrable in Cartan sense but $\mathcal{P}$ is not integrable in Pfaff sense.*

Let $\mathcal{P} = span_{\mathcal{F}(M)}\{\omega = dx - xzdy\}$, $M = \mathbf{R}^3$. We have: $d\omega = -zdx \wedge dy - xdz \wedge dy$ and $d\omega \wedge \omega = xdx \wedge dy \wedge dz \not\equiv 0$. It follows that $\mathcal{P}$ is not integrable in Pfaff sense. But: $N_0 = \{(0, y, z); y, z \in \mathbf{R}\}$ is an integral manifold of P, passing through the origin. $\diamond$

Remark This example is the dual of Example 2.7 (see [Ba92] or [Fr78]) that shows that it is not necessary to have involutive module of smooth vector field for obtain punctual integrability.

Let $\mathcal{P} = span_{\mathcal{F}(M)}\{\omega_1, \cdots, \omega_q\}$ be a differential system. By searching a dual definition of involutivity for differential systems we can try the following conditions that we call Frobenius' conditions:

$$(F_1) \;\; \forall \omega \in \mathcal{P}, \exists \pi_1, \ldots, \pi_q \in \Lambda^1(M) \mid d\omega = \sum_{i=1}^{q} \pi_i \wedge \omega_i$$

$$(F_2) \;\; \forall \omega \in \mathcal{P}, \; d\omega \wedge \omega_1 \wedge \ldots \wedge \omega_q = 0$$

If P is a codistribution without singularities (a codistribution with constant rank) then we can choose for the set of generators of $\mathcal{P}$ exactly $q = \dim P$ elements and an algebraic result says that F_1 and F_2 are equivalent conditions. In this case (of codistribution with constant rank) the problem of integrability is solved by the classical Frobenius theorem:

THEOREM 2.3 *If P is a C^r codistribution without singularities then the following conditions are equivalent:*

(a) P is locally integrable (that means in Cartan sense at every point); (b) $\mathcal{P}$ is integrable in Pfaff sense; (c) F_1 or F_2 (because they are equivalent). $\square$

In the general case of the codistributions with singularities, the following implications are obvious:

PROPOSITION 2.4 *$\mathcal{P}$ is integrable in Pfaff sense $\Longrightarrow F_1 \Longrightarrow F_2$.* □

A converse of this result is given in Malgrange's papers (see [Ma76, Ma77]) where are imposed supplementary conditions about the codimension of the set of singularities.

For obtain integrability in Cartan sense, we will start by constructing a few objects associated to the differential system and then we can impose the dual condition of involutivity (to transfer results from our previous paper about distributions to the codistributions).

3 Construction of objects

Let $\mathcal{P}$ be a $\mathcal{F}(M)$-module of $\mathcal{C}^r$ 1-forms and let P denote the associated distribution. Let $x_0 \in M$ be a fixed point. Let $n = \dim M$ and $k = n - \dim P(x_0) = n - \dim \mathcal{P}|_{x_0}$.

A vector field $X \in V^r(M)$ will be called an *elementary vector field* if there exist $n-k$ Pfaff forms in $\mathcal{P} : \omega_{k+1}, \ldots, \omega_n \in \mathcal{P}$ such that:
1) $\omega_{k+1}|_{x_0}, \ldots, \omega_n|_{x_0}$ are independent;
2) $\omega_j(X) = 0$, $j = \overline{k+1, n}$
where $\omega_j(X) = i_X\omega_j = X\rfloor\omega_j$ and denotes the inner product.

We denote by $T\mathcal{P}$ the $\mathcal{F}(M)$-module generated by the elementary vector fields and we call it the *tangent module* of $\mathcal{P}$ at x_0. By convention, if $k = n$ then $T\mathcal{P} = V^r(M)$.
Remark *The set of elementary vector fields is not closed under the addition operation (that means that there exist two elementary vector fields (X_1, X_2) such that $X_1 + X_2$ is not an elementary vector field).*

We associate an ideal of functions to the differential system:

$$I\mathcal{P} \stackrel{\text{def}}{=} \{\omega(x); \omega \in \mathcal{P}, X \in T\mathcal{P}\}$$

It is an ideal in the ring $\mathcal{F}(M)$ of all $\mathcal{C}^r$ real-valued functions.

We call the *annihilator* of $\mathcal{P}$ the $\mathcal{F}(M)$-submodule of $V^r(M)$:

$$\mathcal{P}^\perp \stackrel{\text{def}}{=} \{X \in V^r(M) | \omega(X) = 0,\ \forall\omega \in \mathcal{P}\}$$

We introduce now the *derived system* of $\mathcal{P}$:

$$D\mathcal{P} \stackrel{\text{def}}{=} \mathcal{P} + span_{\mathcal{F}(M)}\{L_X\omega; X \in T\mathcal{P} \ and \ \omega \in \mathcal{P}\}$$

where $L_X\omega$ denotes the Lie derivative of the Pfaff form ω with respect to the vector field X. Inductively we put: $D^{k+1}\mathcal{P} \stackrel{\text{def}}{=} D(D^k\mathcal{P})$, where $D^0\mathcal{P} = \mathcal{P}$ and by convention: $D^\infty\mathcal{P} = \bigcup_{k\geq 0} D^k\mathcal{P}$.

We choose a neighborhood $\mathcal{U}$ of x_0 and a set of generators (in this neighborhood) of the module of the following form:

$$B = \left(\begin{array}{ccc|c} g^i_\beta & & 1 \leq i \leq k & 0 \\ & & \beta \in I & \\ \hline \omega^i_j & & 1 \leq i \leq k & I_{n-k} \\ & & k+1 \leq j \leq n & \end{array} \right) \tag{1}$$

where I is an index set and g^i_β, ω^i_j are the components of the Pfaff forms that span $\mathcal{P}$ and ordered according to the row ($g^i_\beta(x_0) = 0$). From now on we will agree implicitly that $x \in \mathcal{U}$ and every object is restricted on $\mathcal{U}$.

The connections among the previous notions are given by the following proposition that we give without proof (for a proof see [Fr84]).

PROPOSITION 3.1 *a) The module $T\mathcal{P}$ is generated by the vector fields:*

$$a_i = \frac{\partial}{\partial x^i} - \sum_{j=k+1}^{n} \omega^i_j \frac{\partial}{\partial x^j} \ ,$$

$$a_{ij\beta} = g^i_\beta \frac{\partial}{\partial x^j}$$

where $1 \leq i \leq k$ $k+1 \leq j \leq n$ and $\beta \in I$
b) The ideal $I\mathcal{P}$ is generated by $\{g^i_\beta\}$: $I\mathcal{P} = (g^i_\beta)$.
c) $\mathcal{P}^\perp \subset T\mathcal{P}$
d) Let $\mathcal{P}$ and $\mathcal{J}$ denote two differential systems such that $\mathcal{P}|_{x_0} = \mathcal{J}|_{x_0}$. Then: $\mathcal{P}^\perp \subset \mathcal{J}^\perp \subset T\mathcal{J} \subset T\mathcal{P}$.
e) If $\mathcal{P}|_{x_0} \neq \{0\}$ (i.e. $k < n$) then:

$$D\mathcal{P} = \mathcal{P} + d(I\mathcal{P}) + i_{T\mathcal{P}}(d\mathcal{P})$$

where: $d(I\mathcal{P}) = \{df | f \in I\mathcal{P}\}$ and $i_{T\mathcal{P}}(d\mathcal{P}) = \{i_X d\omega | X \in T\mathcal{P}, \omega \in \mathcal{P}\}$
($i_X d\omega = X \rfloor d\omega$ denotes the inner product).
f) $D(D^\infty \mathcal{P}) = D^\infty \mathcal{P}$ $\square$

The previous construction is entirely due to Freeman. We complete this by introducing two structures:

$$F_{(-1)} \stackrel{\text{def}}{=} span_{\mathbf{R}}\{\omega_j | \omega_j = \sum_{i=1}^{k} \omega^i_j dx_i + dx_j \ ; \ k+1 \leq j \leq n\} = \{\sum_{j=k+1}^{n} a_j \omega_j \ ; \ a_j \in \mathbf{R}\}$$

$$\mathcal{H} \stackrel{\text{def}}{=} \{\omega \in \mathcal{P}|_{\mathcal{U}}, \ \omega = \sum_{i=1}^{k} \omega^i dx_i\} = \mathcal{H}|_{\mathcal{U}} = span_{\mathcal{F}(\mathcal{U})}\{\sum_{i=1}^{k} g^i_\beta dx_i | \beta \in I\}$$

From the dimensional relation: $\dim F_{(-1)}|_{x_0} = \dim P(x_0)$ results $\mathcal{H}|_{x_0} = \{0\}$.
It is very easy to prove the following lemma:

LEMMA 3.2 *The codistribution generated by $\mathcal{H} \oplus F_{(-1)}$ coincides locally with P. That means: $P(x) = \mathcal{H}|_x \oplus F_{(-1)}|_x$, for all $x \in \mathcal{U}$ ($\oplus$ denotes the direct sum).* $\square$

We have obtained two algebraic structures which generates locally the codistribution: $F_{(-1)}$, which is a $(n-k)$-dimensional $\mathbf{R}$-vector subspace and $\mathcal{H}$, which is a $\mathcal{F}(\mathcal{U})$-module. We say that $(F_{(-1)}, \mathcal{H})$ is a *split of codistribution* generated by $\mathcal{P}$.

We define:

$$L_{(+1)} \stackrel{\text{def}}{=} span_{\mathbf{R}}\{a_i | a_i = \frac{\partial}{\partial x^i} - \sum_{j=k+1}^{n} \omega_j^i \frac{\partial}{\partial x^j} \ , \ i = \overline{1,k}, j = \overline{k+1,n}\}$$

which is a k-dimensional $\mathbf{R}$-vector subspace orthogonal to $F_{(-1)}$ (with respect to the inner product). Obvious: $L_{(+1)}|_{x_0} = T\mathcal{P}|_{x_0}$.

We put:

$$\mathcal{N}_{\varepsilon.x_0} \stackrel{\text{def}}{=} \{\exp(\sum_{i=1}^{k} \alpha_i v_i.x_0) | \sum_{i=1}^{k} |\alpha_i| < \varepsilon\}$$

for an $\varepsilon > 0$ small enough, and denote by:

$$\mathcal{G} \stackrel{\text{def}}{=} span_{\mathcal{F}(\mathcal{U})}\{v_j | v_j = g_\beta^i \frac{\partial}{\partial x^j} | \ 1 \le i \le k, k+1 \le j \le n, \beta \in I\}$$

a $\mathcal{F}(\mathcal{U})$-module. We observe that $(L_{(+1)}, \mathcal{G})$ is a split of distribution generated by $T\mathcal{P}$ (see [Ba92]).

The following lemma results directly from definition of integrability and construction of the tangent module (see also [Fr84]):

LEMMA 3.3 *If $\mathcal{P}|_{x_0} \neq \{0\}$ then the codistribution P is integrable in Cartan sense at x_0 if and only if the distribution generated by $T\mathcal{P}$ is punctually integrable at x_0.* $\square$

We will apply all results about integrability of distributions from our previous paper to the module $T\mathcal{P}$.

Now we search a condition equivalent to involutivity of distributions wich enables us to obtain a reciproc of Theorem 4.4 from [Ba92]. Example 2.1 shows that the conditions F_1 or F_2 are not sufficient for integrability in Cartan sense. We say that $\mathcal{P}$ is *involutive* if:

$$(In) \qquad D\mathcal{P} = \mathcal{P}$$

We remark that in the case of codistributions without singularities all the three conditions (F_1, F_2 and In) are equivalent. The meaning of the definition is given by the following result:

PROPOSITION 3.4 *If $\mathcal{P}$ is involutive then $T\mathcal{P}$ is involutive.*

Proof

We use the form given by Proposition 3.1.
Let $\omega_\beta = \sum_{i=1}^k g_\beta^i dx_i \in \mathcal{P}$. *From* $L_v\omega_\beta = \sum_{i=1}^k (L_v g_\beta^i) dx_i \in \mathcal{P}$ *we conclude that:* $[v, g_\beta^i \frac{\partial}{\partial x^j}] \in T\mathcal{P}$, *for all* $v \in T\mathcal{P}$.
We have also:

$$(L_{a_i}\omega_j)(a_s) = a_i(\underbrace{\omega_j(a_s)}_{=0}) - \omega_j([a_i, a_s]) = \omega_j([a_s, a_i])$$

$$(L_{a_i}\omega_j)(a_s) = (\sum_{\alpha=1}^k f_{ij\alpha}\omega_\alpha + \sum_{\beta\in I} f_{ij\beta}\omega_\beta)(a_s) = \sum_{\beta\in I} f_{ij\beta} g_\beta^s$$

where $1 \le i, s \le k$. *Then:*

$$[a_s, a_i] = \sum_{j=k+1}^n \omega_j([a_s, a_i])\frac{\partial}{\partial x^j} = \sum_{j=k+1}^n \sum_{\beta\in I} f_{ij\beta} g_\beta^s \frac{\partial}{\partial x^j} \in T\mathcal{P}$$

Q.E.D. □

Remarks

1. The reciproc of Proposition 3.4 is not true. For example let $\mathcal{P} = span_{\mathcal{F}(M)}\{ydx\}$ in $\mathbf{R}^2$. Since $T\mathcal{P} = V^r(\mathbf{R}^2)$ we conclude that $T\mathcal{P}$ is involutive but $L_{\frac{\partial}{\partial y}}(ydx) = dx \notin \mathcal{P}$. So $\mathcal{P}$ is not involutive. ◇

2. We have also another example: $M = \mathbf{R}^4, \mathcal{P} = span_{\mathcal{F}(M)}\{x^3dx_1 + x^4dx_2, dx_3, dx_4\}$ and $x_0 = (0,0,0,0)$ (the origin). We obtain:

$$T\mathcal{P} = span_{\mathcal{F}(M)}\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, x^3\frac{\partial}{\partial x^3}, x^3\frac{\partial}{\partial x^4}, x^4\frac{\partial}{\partial x^3}, x^4\frac{\partial}{\partial x^4}\}$$

which is involutive. But $L_{x^1\frac{\partial}{\partial x^1}}(x^3dx_1 + x^4dx_2) = x^3dx_1 \notin \mathcal{P}$.

4 Punctual results

We remark that $(L_{(+1)}, \mathcal{G})$ is a split of the distribution generated by $T\mathcal{P}$(*see* [Ba92]). We obtain very easy the following result that is dual of Proposition 4.1 from [Ba92]:

PROPOSITION 4.1 *The codistribution* P *is integrable in Cartan sense at* x_0 *if and only if:*
1) $F_{(-1)}|_x = (T_x\mathcal{N}_{\varepsilon.x_0})^\perp$, $\forall x \in \mathcal{N}_{\varepsilon.x_0}$
2) $\mathcal{H}|_{\mathcal{N}_{\varepsilon.x_0}} = 0$ *(that means* $\mathcal{H}|_x = 0$ *for all* $x \in \mathcal{N}_{\varepsilon.x_0}$*).*
In this case $\mathcal{N}_{\varepsilon.x_0}$ *is an integral manifold of the codistribution passing through* x_0. □

Remark Equivalent to 2 is the condition: $2')$ $I\mathcal{P}|_{\mathcal{N}_{\varepsilon.x_0}} = 0$.

Let $u, v \in L_{(+1)}$ and $\omega \in F_{(-1)}$. We have:

$$d\omega(u,v) = u(\omega(v)) - v(\omega(u)) - \omega([u,v]) = -\omega([u,v])$$

and:

$$(L_u\omega)(v) = u(\omega(v)) - \omega([u,v]) = -\omega([u,v])$$

So, if we require that $d\omega(u,v)|_x = 0$ or $(L_u\omega)(v)|_x = 0$ for all $\omega \in F_{(-1)}$ and $v \in L_{(+1)}$, we obtain $[u,v]|_x = 0$. Then the dual version of Corollary 4.3 (from [Ba92]) and also a corollary of Proposition 4.1 is the following:

COROLLARY 4.2 *The codistribution P is integrable in Cartan sense at x_0 if and only if:*

1) $d\omega(u,v)|_{\exp tv.x_0} = 0$, for all $v \in L_{(+1)}$, $\omega \in F_{(-1)}$ and $|t| < \varepsilon$, ε depending on v.

or equivalent:

1') $L_v\omega|_{\exp tv.x_0} \in P(\exp tv.x_0)$, for all $v \in L_{(+1)}, \omega \in F_{(-1)}$ and $|t| < \varepsilon$, ε depending on v.

and

2) $\mathcal{H}|_{\mathcal{N}_{\varepsilon.x_0}} = 0$. □

THEOREM 4.3 *Let $\mathcal{P}$ be an analytic $\mathcal{F}(M)$-module of 1-forms and let P denote the associated codistributions. Then P is integrable in Cartan sense at x_0 if and only if $D^\infty\mathcal{P}|_{x_0} = P(x_0)$.*

Proof

" $\Rightarrow$ " Let $\tilde{\mathcal{N}}_{x_0}$ denote an integral manifold. We shall prove by induction that $D^k\mathcal{P}|_x = P(x)$ for all $x \in \tilde{\mathcal{N}}_{x_0}$. Then we can conclude that $D^\infty\mathcal{P}|_{x_0} = P(x_0)$. For $k = 0$ is obvious. We have $D^k\mathcal{P}|_x = P(x)$ for all $x \in \tilde{\mathcal{N}}_{x_0}$ and using Lemma 3.3 and Nagano's theorem (see [Na66]) we obtain: $TD^k\mathcal{P}|_x = T_x\tilde{\mathcal{N}}_{x_0}$, $\forall x \in \tilde{\mathcal{N}}_{x_0}$. Then: $L_X\omega|_x \in (T_x\tilde{\mathcal{N}}_{x_0})^\perp = P(x)$, for all $X \in TD^k\mathcal{P}$, $\omega \in D^k\mathcal{P}$ and $x \in \tilde{\mathcal{N}}_{x_0}$. So: $D^{k+1}\mathcal{P}|_x = P(x)$, for all $x \in \tilde{\mathcal{N}}_{x_0}$.

" $\Leftarrow$ " It is sufficiently to consider the following sequence:

$$Q^0 = \mathcal{P},\ Q^{k+1} = Q^k + span_{\mathcal{F}(M)}\{L_X\omega,\ X \in L_{(+1)} \text{ and } \omega \in Q^k\}$$

Since $Q^k \subset D^k\mathcal{P}$ we conclude $Q^\infty|_{x_0} = P(x_0) = Q^0|_{x_0}$. We consider the split $(F_{(-1)}, \mathcal{H})$ of codistribution as in §3.

a) Let $\omega \in \mathcal{H}$, $\omega = \sum_{i=1}^k g^i dx_i$; $g^i \in I\mathcal{P}$. By a simple checking: $L_X\omega = \sum_{i=1}^k (L_X g^i)dx_i$, $\forall x \in L_{(+1)}$. Since $Q^\infty|_{x_0} = P(x_0)$ we obtain $L_X^k\omega|_{x_0} = 0$ and so: $g^i|_{\mathcal{N}_{\varepsilon.x_0}} = 0 \Leftrightarrow I\mathcal{P}|_{\mathcal{N}_{\varepsilon.x_0}} = 0$.

b) Let $\omega \in F_{(-1)}$ and $u, v \in L_{(+1)}$. We have: $L_v\omega|_{\exp tv.x_0}(u) = d\omega(v,u)|_{\exp tv.x_0} = 0$ (using Taylor series). Using Corollary 4.2 we obtain the integrability. Q.E.D. □

We can give a criterion of integrability in the smooth case ($r = \infty$) dual of Theorem 4.6 ([Ba92]):

THEOREM 4.4 *Let $\mathcal{P}$ be a $\mathcal{C}^\infty$-differential system and let P denote the codistribution generated. Let $x_0 \in M$ be a fixed point so that $P(x_0) \neq \{0\}$. Let $k = codim P(x_0) = n - \dim P(x_0)$. Then P is integrable in Cartan sense at x_0 if and only if there exist an $\varepsilon > 0$, 1-forms $\omega_{k+1}, \dots, \omega_n \in \mathcal{P}$ and a neighborhood $\mathcal{U}$ of x_0 that satisfy the following conditions:*
1) In the point x_0: $\quad \omega_{k+1}|_{x_0}, \dots, \omega_n|_{x_0}$ span $P(x_0)$.
2) For every smooth vector field $a \in T\mathcal{P}$ there exist smooth functions $\lambda_i^j : (-\mu_a, \mu_a) \to \mathbf{R}$ such that for all $t \in (-\mu_a, \mu_a)$ we have:

$$L_a\omega_i|_{\exp ta.x_0} = \sum_{j=k+1}^{n} \lambda_i^j(t)\omega_j|_{\exp ta.x_0} \tag{2}$$

where: $\mu_a \stackrel{\text{def}}{=} \sup\{\nu | \nu \le \varepsilon$ and $\exp ta.x_0 \in \mathcal{U}$ for all $|t| < \nu\}$

Proof

"$\Rightarrow$ " Let P be integrable in Cartan sense at x_0. We choose $\{\omega_j\}$, ε and $\mathcal{U}$ as in §3.
1) It is checked by construction of the 1-forms.
2) Let $a \in T\mathcal{P}$. Then $a = \sum_{j=1}^k f_j a_j + b$ where a_j are as in the definition of $L_{(+1)}$, $b \in \mathcal{G}$ and $f_j \in \mathcal{F}(\mathcal{U})$. We put: $x_t = \exp ta.x_0$. Since $b|_{x_t} = 0$ from Corollary 4.2 we obtain:

$$\begin{aligned} L_a\omega_i|_{x_t} &= L_{(\sum_{j=1}^k f_j a_j + b)}\omega_i|_{x_t} = L_{\sum_{j=1}^k f_j a_j}\omega_i|_{x_t} \\ &= \sum_{j=1}^k f_j \underbrace{L_{a_j}\omega_i|_{x_t}}_{\in P(x_t)} + \sum_{j=1}^k \underbrace{\omega_i(a_j)}_{=0} df_j|_{x_t} = \sum_{j=k+1}^n \lambda_i^j(t)\omega_j|_{x_t} \end{aligned}$$

"$\Leftarrow$ " First we see that the relation (2) is invariant under the change of the set $\{\omega_j\}$ (we have a result as Lemma 4.5 from [Ba92]). We consider for $\{\omega_i\}$ the same 1-forms as in §3. We choose $a \in L_{(+1)}$ and we obtain $1'$ from Corollary (4.2). We have to prove that $\mathcal{H}|_{\exp ta.x_0} = 0$. Let $a = v + b$ where $v \in F_{(+1)}$ and $b = g_\beta \frac{\partial}{\partial x^j}$; $\beta \in I$ and $k+1 \le j \le n$. We have:

$$(L_a\omega_j)(v)|_{x_t} = 0 = \underbrace{(L_v\omega_j)(v)|_{x_t}}_{=0} + \omega_j([v,b])|_{x_t}$$

and $(L_a\omega_j)(b)|_{x_t} = L_a(\omega_j)(b))|_{x_t} - \omega_j([v,b])|_{x_t}$. With relation (2): $(L_a\omega_j)(b)|_{x_t} = \lambda_j^j(t) \cdot g_\beta|_{x_t}$.
So: $L_a g_\beta|_{x_t} = \lambda_j^j(t) \cdot g_\beta|_{x_t}$. Since $g_\beta(x_0) = 0$, from the theorem of unicity of the solution of the Cauch problem we obtain: $g_\beta(x_t) = 0$. But $x_t = \exp t(v+b).x_0 = \exp tv.x_0$. So: $g_\beta|_{\mathcal{N}_\varepsilon.x_0} = 0$. Q.E.D. □

5 Local results

Now we give a new proof of the main theorem of [Fr84].

THEOREM 5.1 *Let $\mathcal{P}$ be a finitely-generated $\mathcal{C}^\infty$ $\mathcal{F}_{x_0}$-module of germs of* 1-*forms at x_0 and $n-k=\dim\mathcal{P}|_{x_0}$ ($\mathcal{F}_{x_0}$ denotes the ring of germs of smooth real-valued functions at x_0). Then $\mathcal{P}$ is involutive if and only if there exist a coordinate system $(y^1,\ldots,y^n)$ and a finite number of germs of functions $g_\beta(\beta\in I$ and I a finite set of indices) depending only on the last $n-k$ coordinates ($g_\beta=g_\beta(y^{k+1},\ldots,y^n)$) and vanishing at x_0, such that:*

$$\mathcal{P}=span_{\mathcal{F}_{x_0}}\{dx_{k+1},\ldots,dx_n,g_\beta dx_1,\ldots,g_\beta dx_k\mid\beta\in I\}$$

Proof

"$\Leftarrow$ " It is very easy to prove in this sense.
"$\Rightarrow$ "
1. From Proposition 3.4 *we obtain that $T\mathcal{P}$ is an involutive, finitely-generated module of germs of smooth vector fields .*
2. We apply Theorem 5.4 *from [Ba92] and obtain a system of generators for $T\mathcal{P}$ of the form:*

$$\frac{\partial}{\partial y^1},\ldots,\frac{\partial}{\partial y^k},g_\beta\frac{\partial}{\partial y^{k+1}},\ldots,g_\beta\frac{\partial}{\partial y^n}$$

where $g_\beta=g_\beta(y^{k+1},\ldots,y^n)$; $g_\beta(x_0)=0$; $\beta\in I$.
3. Let $\omega_i=\sum_{j=1}^k\omega_i^j dy_j+dy_i$ as in §3. *Since:*

$$L_{y^j\frac{\partial}{\partial y^j}}\omega_i=y^jL_{\frac{\partial}{\partial y^j}}\omega_i+\underbrace{\omega_i(\frac{\partial}{\partial y^j})}_{=\omega_i^j}dy_j$$

we conclude that $\omega_i^j dy_j\in\mathcal{P}$, for all $1\le j\le k$. Then $\omega_i-\sum_{j=1}^k\omega_i^j dy_j\in\mathcal{P}$. That means $dy_i\in\mathcal{P}$; $1\le i\le k$.
4. For obtain the generators $g_\beta\frac{\partial}{\partial y^i}$ of $T\mathcal{P}$ we must have also generators of $\mathcal{P}$ of the form: $\sum_{i=1}^k g_\beta^i dy_i$ where $g_\beta^i=g_\beta^i(y^{k+1},\ldots,y^n)$. Then:

$$L_{y^j\frac{\partial}{\partial y^j}}(\sum_{i=1}^k g_\beta^i dy_i)=\sum_{i=1}^k g_\beta^i L_{y^j\frac{\partial}{\partial y^j}}dy_i=g_\beta^j dy_j\in\mathcal{P};\ 1\le j\le k$$

So: $g_\beta dy_j\in\mathcal{P}$ and the proof is complete. Q.E.D. □

Remarks 1) An integral manifold of $\mathcal{P}$ that passes through x_0 is given by:

$$N_{x_0}=\{y\in M|y^{k+1}=y_0^{k+1},\ldots,y^n=y_0^n\}.$$

2) We see also that $\mathcal{P}$ is integrable in Pfaff sense around x_0. A set of generators of exact forms is given by:

$$\{d(y^1g_\beta),\ldots,d(y^kg_\beta),dy_{k+1},\ldots,dy_n,\ \beta\in I\}$$

COROLLARY 5.2 ([Fr84]) *If $D^{\infty}\mathcal{P}$ is finitely-generated then it is integrable in Cartan sense at x_0 and Pfaff sense around x_0.*

COROLLARY 5.3 ([Fr84]) *If $D^{\infty}\mathcal{P}$ is finitely-generated then P is integrable in Cartan sense at x_0 if and only if: $D^{\infty}\mathcal{P}|_{x_0} = P(x_0)$.*

6 Discussion

We want to point out one type of result arising from our studies. Let $\mathcal{O}b$ denote a $\mathcal{C}^r$-module of vector fields or 1-forms and let x_0 be a fixed point. To this object we associate a module of smooth vector fields: $T\mathcal{O}b$. We carry on the iterative sequence:

$$\mathcal{O}b^{k+1} \stackrel{\text{def}}{=} \mathcal{O}b^k + span_{\mathcal{F}(M)} L_{T\mathcal{O}b}\mathcal{O}b^k, \ for\ k \geq 0,\ \mathcal{O}b^0 = \mathcal{O}b$$

($L_{T\mathcal{O}b}\mathcal{O}b^k \stackrel{\text{def}}{=} \{L_X Y | X \in T\mathcal{O}b,\ Y \in \mathcal{O}b^k\}$ and $\mathcal{O}b^{\infty} \stackrel{\text{not}}{=} \cup_{k\geq 0}\mathcal{O}b^k$).

A local result is: If $\mathcal{O}b$ is finitely generated and $\mathcal{O}b^1 = \mathcal{O}b^0 = \mathcal{O}b$ then there exist a system of coordinates such that $\mathcal{O}b$ has a normal form and the (co)distribution is punctually integrable at x_0.

A punctual result is: If $\mathcal{O}b^{\infty}$ is finitely generated (that is happened, for example, in the analytic case) then the (co)distribution associated is punctually integrable at x_0 if and only if: $\mathcal{O}b^{\infty}|_{x_0} = \mathcal{O}b|_{x_0}$.

We see that in the case of distributions $T\mathcal{O}b = \mathcal{O}b$ and in the case of codistributions $T\mathcal{O}b$ is exactly the tangent module.

This technique may be applied also at the systems of smooth k-forms.

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