Excesses of Gabor Frames

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Abstract

A Gabor system for $L^2(\mathbf{R}^d)$ has the form $\mathcal{G}(g,\Lambda) = \{e^{2\pi i b\cdot x} g(x-a)\}_{(a,b)\in\Lambda}$, where $g \in L^2(\mathbf{R}^d)$ and Λ is a sequence of points in $\mathbf{R}^{2d}$. We prove that, with only a mild restriction on the generator g and for nearly arbitrary sets of time-frequency shifts Λ, an overcomplete Gabor frame has infinite excess, and in fact there exists an infinite subset that can be removed yet leave a frame. The proof of this result yields an interesting connection between the density of Λ and the excess of the frame.

Key words: Density, excess, frames, Gabor systems, modulation spaces, Riesz bases, wavelets, Weyl–Heisenberg systems.

1 Introduction

A countable sequence $\mathcal{F} = \{f_i\}_{i\in I}$ of elements of a Hilbert space H is a *frame* for H if there exist constants A, $B > 0$ (called *frame bounds*) such that

$$\forall\, h \in H, \quad A\,\|h\|^2 \;\le\; \sum_{i\in I} |\langle h, f_i\rangle|^2 \;\le\; B\,\|h\|^2. \tag{1}$$

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The frame is *tight* if we can take $A = B$. It is a *normalized tight frame* or a *Parseval frame* if we can take $A = B = 1$.

Frames were first introduced by Duffin and Schaeffer (8) in the context of non-harmonic Fourier series, and have since seen a wide variety of applications in science, mathematics, and engineering. The *frame operator* $Sh = \sum_{i\in I}\langle h, f_i\rangle f_i$ is a positive, continuous mapping of H onto itself with continuous inverse. The frame $\mathcal{F}$ together with its *standard dual frame* $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I} = \{S^{-1}f_i\}_{i\in I}$ provides the *frame expansions*

$$h \;=\; \sum_{i\in I}\langle h, \tilde{f}_i\rangle\, f_i \;=\; \sum_{i\in I}\langle h, f_i\rangle\, \tilde{f}_i. \tag{2}$$

However, these representations need not be unique, i.e., $\mathcal{F}$ need not be a basis. In fact, $\mathcal{F}$ is a basis if and only if it is a Riesz basis. We say that a frame that is not a basis is *overcomplete* or *redundant*. For each $j \in I$, $\mathcal{F}\setminus\{f_j\} = \{f_i\}_{i\neq j}$ is either incomplete or is itself a frame. The *excess* of $\mathcal{F}$, denoted $e(\mathcal{F})$, is the supremum of the cardinalities of all subsets $J \subset I$ such that $\{f_i\}_{i\in I\setminus J}$ is complete in H.

The prior paper (1) studied the excess of frames. Among other results, it was shown that the supremum in the definition of excess is achieved, so in particular if $e(\mathcal{F}) = \infty$ then there is an infinite subset $J \subset I$ such that $\{f_i\}_{i\in I\setminus J}$ is complete. However, it need not be true that $\{f_i\}_{i\in I\setminus J}$ is a frame. In fact, (1, Example 5.1) is an example of a Parseval frame with infinite excess such that $\{f_i\}_{i\in I\setminus J}$ is not a frame for *any* infinite subset J of I. Several characterizations of when there does exist an infinite subset J such that $\{f_i\}_{i\in I\setminus J}$ is a frame were obtained in (1), quoted in Theorem 6 below and extended in Theorem 9.

In this paper we are concerned with the special case of *Gabor frames* for the Hilbert space $L^2(\mathbf{R}^d)$. For $x,\ \omega \in \mathbf{R}^d$, let $T_x f(t) = f(t-x)$ and $M_\omega f(t) = e^{2\pi i\omega\cdot t} f(t)$ denote the unitary operators of translation and modulation. Then we define a *time-frequency shift* of a function f on $\mathbf{R}^d$ by $z = (x,\omega) \in \mathbf{R}^{2d} = \mathbf{R}^d \times \mathbf{R}^d$ to be

$$\pi(z)f(t) \;=\; M_\omega T_x f(t) \;=\; e^{2\pi i\omega\cdot t}\, f(t-x). \tag{3}$$

Given a fixed *window function* $g \in L^2(\mathbf{R}^d)$ and given a sequence Λ of points in $\mathbf{R}^{2d}$ (repetitions are allowed), the *Gabor system* generated by g and Λ is

$$\mathcal{G}(g,\Lambda) \;=\; \{\pi(\lambda)g\}_{\lambda\in\Lambda}. \tag{4}$$

A Gabor system which is a frame is called a Gabor frame. We refer to (7; 17; 20) for background information on Gabor and other frames.

Understanding the redundancy of frames, and of Gabor frames in particular, is a fundamental issue that has impact on many applications. For example, in multiple description encoding schemes, coefficients may be lost due to channel erasure. An analysis of such encoders using Gabor frames was obtained in (2). The problem of erasures in the setting of finite dimensional frames was considered by Goyal, Kovačević and Kelner in (16) and comprehensively analyzed in (4). In (9), Eldar and Bölcskei studied the impact of removing single or multiple elements from unitarily generated frames, and obtained estimates of the frame bounds for such frames.

In this paper we consider the redundancy of Gabor frames with respect to arbitrary sequences of time-frequency shifts Λ. We prove, assuming some slight restrictions on the generator g and on the set Λ, that any Gabor frame which is not a Riesz basis has infinite excess, and furthermore that an infinite subset of the frame may be removed yet leave a frame.

Very few theoretical results are available for the "irregular" Gabor systems considered in this paper. Most results for Gabor systems require a structural assumption on Λ, usually that $\Lambda = \alpha\mathbf{Z}^d \times \beta\mathbf{Z}^d$ (a "rectangular lattice"). However, Gabor systems with respect to other sequences Λ of time-frequency shifts arise naturally, for example by perturbations of a regular system or directly from the constraints of an application, as in (25), where a Gabor frame with a non-rectangular lattice Λ is applied to wireless coding. It is shown in (22) that Gabor systems which are orthonormal bases for $L^2(\mathbf{R}^d)$ can exist even with completely aperiodic Λ.

Two important results that are available for irregular Gabor systems are the density theorems of Ramanathan and Steger (23) and of Janssen (21). The result of Ramanathan and Steger (as extended in (5)) is as follows.

Theorem 1 (Density Theorem)

a. *If* $\mathcal{G}(g,\Lambda)$ *is a frame for* $L^2(\mathbf{R}^d)$*, then* $1 \le D^-(\Lambda) \le D^+(\Lambda) < \infty$.
b. *If* $\mathcal{G}(g,\Lambda)$ *is a Riesz basis for* $L^2(\mathbf{R}^d)$*, then* $D^-(\Lambda) = D^+(\Lambda) = 1$.

Here $D^{\pm}(\Lambda)$ are the upper and lower Beurling densities of Λ, which measure in some sense the average number of elements of Λ lying inside sets of unit measure (defined precisely in Section 2.1). To prove Theorem 1, Ramanathan and Steger showed that each Gabor frame satisfies a certain Homogeneous Approximation Property (HAP). This HAP seems to be of independent interest, yet, so far as we are aware, no application of it to results other than density conditions has been made.

Janssen has obtained in (21) another density result, for the case of generalized Gabor systems in $L^2(\mathbf{R})$ of the form $\{g_m(x-na)\}_{m,n\in\mathbf{Z}}$. Assuming that each

g_m is localized in frequency around a point b_m, and given some simultaneous control on the decay of the functions g_m in the frequency domain, Janssen obtained a necessary condition on the density of the set $\{(na, b_m)\}_{m,n\in\mathbf{Z}}$ in order that $\{g_m(x - na)\}_{m,n\in\mathbf{Z}}$ be a frame. Although we will not pursue this type of generalization here, it is an interesting topic for future work.

Our first main result is the following theorem (Theorem 2). It states that, with only a mild restriction on g and the assumption that $D^+(\Lambda) > 1$, there is a fundamental connection between a certain quantity directly tied to the excess of a Gabor frame and the density of that frame. Moreover, the HAP plays an important role in the proof. An immediate consequence of this relationship is that not only is the excess of the Gabor frame infinite, but there exists an infinite subset that can be removed yet still leave a frame. The form of the inequality (5) in Theorem 2 suggests the potential for additional insights into frame theory in general by examining trace-like features of the projection operator associated to a frame.

The modulation space M^p appearing in the statement of the following theorem is defined precisely in Section 2.4. We remark here only that membership in M^p corresponds to a certain amount of joint localization in both time and frequency, that M^p is dense in L^2 for $p < 2$, and that $M^2 = L^2$. The set $I(r, z)$ appearing in the statement of the theorem is the intersection of Λ with the cube $Q(r, z)$ centered at z with side lengths r.

Theorem 2 *Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame for $L^2(\mathbf{R}^d)$. If $g \in \bigcup_{1\le p<2} M^p$, then*

$$\liminf_{r\to\infty} \inf_{z\in\mathbf{R}^{2d}} \frac{1}{|I(r, z)|} \sum_{\lambda\in I(r,z)} \langle g_\lambda, \tilde{g}_\lambda\rangle \;\le\; \frac{1}{D^+(\Lambda)}. \tag{5}$$

Consequently, if $D^+(\Lambda) > 1$ then there exists an infinite subset J of Λ such that $\mathcal{G}(g, \Lambda\backslash J)$ is a frame for $L^2(\mathbf{R}^d)$.

If $\mathcal{G}(g, \Lambda)$ is an overcomplete frame and Λ is a *lattice* in $\mathbf{R}^{2d}$, then necessarily $D^+(\Lambda) > 1$ (a lattice is the image of $\mathbf{Z}^{2d}$ under an invertible linear transformation). However, this is not the case when Λ is not a lattice. For example, we can start with a Gabor Riesz basis $\mathcal{G}(g, \Lambda)$, which by the Density Theorem must satisfy $D^-(\Lambda) = D^+(\Lambda) = 1$, and add finitely many points to Λ (or even infinitely many if judiciously chosen) to obtain an overcomplete Gabor frame with the same density. This marginal case is not addressed by Theorem 2.

Theorem 2 can be extended to the case of frames of the form $\mathcal{G}(g_1, \Lambda_1) \cup \cdots \cup \mathcal{G}(g_r, \Lambda_r)$. Our second main result states that in the rectangular lattice setting, i.e., the case where Λ_k has the form $\alpha_k\mathbf{Z}^d \times \beta_k\mathbf{Z}^d$, the assumption in Theorem 2 that g lies in some modulation space M^p can be removed. Additionally, for this result we only need to require that the system be a frame for its closed span, not for the entire space. This result was obtained in (1) for the special case that either $(\alpha_1, \dots, \alpha_r)$ or $(\beta_1, \dots, \beta_r)$ are rationally related, including in particular the case $r = 1$. We present in this paper a new approach that applies even to the irrationally related case.

Theorem 3 *Let $g_1, \dots, g_r \in L^2(\mathbf{R}^d)$, and let $\Lambda_k = \alpha_k \mathbf{Z}^d \times \beta_k \mathbf{Z}^d$ for $k = 1, \dots, r$. If $\mathcal{F} = \mathcal{G}(g_1, \Lambda_1) \cup \cdots \cup \mathcal{G}(g_r, \Lambda_r)$ is an overcomplete frame for its closed span H in $L^2(\mathbf{R}^d)$, then this frame has infinite excess and there exists an infinite subset of $\mathcal{F}$ that can be removed yet leave a frame for H. In fact, this subset can be taken to have the form $\{T_{\alpha_k n_j} g_k\}_{j=1}^{\infty}$, i.e., translates of one of the generators g_k.*

Theorem 3 can be extended to more general lattices by applying a metaplectic transformation, cf. (17, Sec. 9.4) or (18) for background information on this type of extension. In particular, by applying a metaplectic transformation, Theorem 3 can be extended to the case where each Λ_i is a symplectic lattice in $\mathbf{R}^{2d}$ with respect to the same symplectic matrix, i.e., each Λ_i has the form $\Lambda_i = A(\alpha_i \mathbf{Z}^d \times \beta_i \mathbf{Z}^d)$ where A is a fixed $2d \times 2d$ symplectic matrix. When $d = 1$, every lattice in $\mathbf{R}^2$ is a symplectic lattice, but this is not the case when $d > 1$. Specializing to the case $d = 1$ and a single generator therefore yields the following corollary.

Corollary 4 *Let $g \in L^2(\mathbf{R})$, and let Λ be a lattice in $\mathbf{R}^2$. If $\mathcal{F} = \mathcal{G}(g, \Lambda)$ is an overcomplete frame for its closed span H in $L^2(\mathbf{R})$, then this frame has infinite excess and there exists an infinite subset of $\mathcal{F}$ that can be removed yet leave a frame for H.*

The techniques used to prove Theorem 3 can also be applied to the case of wavelets. Given $a > 1$ and $b > 0$, define the wavelet system generated by g, a, b to be

$$\mathcal{W}(g, a, b) \;=\; \{a^{nd/2} g(a^n x - mb)\}_{m \in \mathbf{Z}^d, n \in \mathbf{Z}}. \tag{6}$$

Then the following result can be proved similarly to Theorem 3.

Theorem 5 *Let $g_1, \dots, g_r \in L^2(\mathbf{R}^d)$, let $a_1, \dots, a_r > 1$, and let $b_1, \dots, b_r > 0$ be given. If $\mathcal{F} = \mathcal{W}(g_1, a_1, b_1) \cup \cdots \cup \mathcal{W}(g_r, a_r, b_r)$ is an overcomplete frame for its closed span H in $L^2(\mathbf{R}^d)$, then this frame has infinite excess and there exists an infinite subset of $\mathcal{F}$ that can be removed yet leave a frame for H. In fact, this subset can be taken to consist of dilates of one of the generators g_k.*

2 Preliminaries

2.1 General Notation

Let Λ be a sequence of points in $\mathbf{R}^{2d}$. Then the *lower and upper Beurling densities* of Λ are, respectively,

$$D^-(\Lambda) = \liminf_{r\to\infty} \inf_{z\in\mathbf{R}^{2d}} \frac{|\Lambda \cap Q(r,z)|}{r^{2d}} \tag{7}$$

and

$$D^+(\Lambda) = \limsup_{r\to\infty} \sup_{z\in\mathbf{R}^{2d}} \frac{|\Lambda \cap Q(r,z)|}{r^{2d}}, \tag{8}$$

where $|E|$ denotes the cardinality of a set E, and where $Q(r,z)$ is the cube centered at $z = (z_1, \dots, z_{2d}) \in \mathbf{R}^{2d}$ with side lengths r, i.e.,

$$Q(r,z) = \prod_{i=1}^{2d} [z_i - \frac{r}{2}, z_i + \frac{r}{2}]. \tag{9}$$

2.2 Excess

The following result from (1) will play an important role.

Theorem 6 *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame for a Hilbert space H, with standard dual frame $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$. Then the following statements are equivalent.*

a. *There exists an infinite $J_1 \subset I$ such that $\{f_i\}_{I\setminus J_1}$ is a frame for H.*
b. *There exists $L > 0$ and an infinite $J_2 \subset I$ such that for each $j \in J_2$, $\{f_i\}_{i\neq j}$ is a frame for H with lower frame bound L.*
c. *There exists an infinite $J_3 \subset I$ such that $\sup_{i\in J_3} \langle f_i, \tilde{f}_i\rangle < 1$.*

Remark 7 a. It is easy to see that $0 \le \langle f_i, \tilde{f}_i\rangle \le 1$ for any frame, because $\langle f_i, \tilde{f}_i\rangle = \|S^{-1/2} f_i\|^2$ and $\{S^{-1/2} f_i\}_{i\in I}$ is a Parseval frame for H.

b. Although the sets J_1, J_2, J_3 in Theorem 6 need not coincide in general, we do have $J_1 \subset J_2$.

c. Theorem 6 can be refined to include sharp frame bound estimates, cf. (1) for details.

d. If $T(f) = \{\langle f, f_i\rangle\}_{i\in I}$ is the *analysis operator* for the frame $\mathcal{F}$, then $P = T(T^*T)^{-1}T^*$ is the orthogonal projection of $\ell^2(I)$ onto range(T). The diagonal elements of the matrix representation for P in the standard basis for $\ell^2(I)$ are $\langle f_i, \tilde{f}_i\rangle$.

The following result provides a useful sufficient condition for ensuring that statement c in Theorem 6 will hold.

Lemma 8 *Let $a = (a_i)_{i\in I}$ be a countable sequence of real numbers with $0 \le a_i \le 1$ for each i. If there exist finite subsets I_n of I such that $\lim |I_n| = \infty$ and*

$$\liminf_{n\to\infty} \frac{1}{|I_n|} \sum_{i\in I_n} a_i \; < \; 1, \tag{10}$$

then there is an infinite subset $J \subset I$ such that $\sup\limits_{j\in J} a_j < 1$.

PROOF. Let $r = \liminf_{n\to\infty} \frac{1}{|I_n|} \sum_{i\in I_n} a_i$, and choose s, ε so that $r < s-\varepsilon < s < 1$. Define $F_n = \{i \in I_n : a_i \le s\}$. By (10), there exist $n_k \to \infty$ such that

$$\frac{1}{|I_{n_k}|} \sum_{i\in I_{n_k}} a_i \; \le \; s - \varepsilon. \tag{11}$$

At most $|F_{n_k}|$ terms in the summation on the left side of (11) are smaller than s, so we have

$$s - \varepsilon \; \ge \; \frac{1}{|I_{n_k}|} \sum_{i\in I_{n_k}} a_i \; \ge \; \frac{s\,(|I_{n_k}| - |F_{n_k}|)}{|I_{n_k}|} \; = \; s - \frac{|F_{n_k}|}{|I_{n_k}|}. \tag{12}$$

Hence $|F_{n_k}|/|I_{n_k}| \ge \varepsilon$ for each k. Since $\lim |I_n| = \infty$, it follows that $|\bigcup F_n| = \infty$. □

2.3 Deletions from Frames

In this section we will prove some new results which extend Theorem 6 further.

Theorem 9 *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame for a Hilbert space H, with frame bounds A, B. Let $J \subset I$ be given, and define truncated analysis operators $T_J \colon H \to \ell^2(J)$ and $T_{I\setminus J} \colon H \to \ell^2(I\setminus J)$ by*

$$T_J(f) \; = \; (\langle f, f_i\rangle)_{i\in J} \qquad \text{and} \qquad T_{I\setminus J}(f) \; = \; (\langle f, f_i\rangle)_{i\in I\setminus J}. \tag{13}$$

Then the following statements hold.

a. *If there exists a bounded operator* $L\colon \ell^2(J) \to \ell^2(I\backslash J)$ *such that*

$$\gamma \;=\; \|T_J^* - T_{I\backslash J}^* L\|^2 \;<\; \frac{A}{2}, \tag{14}$$

then $\mathcal{F}' = \{f_i\}_{i\in I\backslash J}$ *is a frame for* H*, with frame bounds* $A' = \frac{A-2\gamma}{1+2\|L\|^2}$, $B' = B$.

b. *If* $\mathcal{F}' = \{f_i\}_{i\in I\backslash J}$ *is a frame for* H*, then there exists a bounded operator* $L\colon \ell^2(J) \to \ell^2(I\backslash J)$ *such that (14) holds with* $\gamma = 0$.

PROOF. a. Assume L satisfies (14). Then

$$\begin{aligned}\|T_J f\|^2 &\le \left(\|T_J f - L^* T_{I\backslash J} f\| + \|L^* T_{I\backslash J} f\|\right)^2 \\ &\le 2\,\|T_J f - L^* T_{I\backslash J} f\|^2 + 2\,\|L^* T_{I\backslash J} f\|^2 \\ &\le 2\gamma\,\|f\|^2 + 2\,\|L\|^2\,\|T_{I\backslash J} f\|^2. \end{aligned} \tag{15}$$

Therefore,

$$\begin{aligned}A\,\|f\|^2 \;\le\; \sum_{i\in I} |\langle f, f_i\rangle|^2 \;&=\; \|T_J f\|^2 + \|T_{I\backslash J} f\|^2 \\ &\le\; 2\gamma\,\|f\|^2 + (1 + 2\,\|L\|^2)\,\|T_{I\backslash J} f\|^2. \end{aligned} \tag{16}$$

Consequently,

$$A'\,\|f\|^2 \;=\; \frac{A - 2\gamma}{1 + 2\|L\|^2}\,\|f\|^2 \;\le\; \|T_{I\backslash J} f\|^2 \;=\; \sum_{i\in I\backslash J} |\langle f, f_i\rangle|^2, \tag{17}$$

which establishes that $\mathcal{F}'$ has a lower frame bound of A'. The upper frame bound is trivial since $\mathcal{F}'$ is a subset of $\mathcal{F}$.

b. Assume $\mathcal{F}' = \{f_i\}_{i\in I\backslash J}$ is a frame for H, and let $\tilde{\mathcal{F}}'$ be the standard dual frame of $\mathcal{F}'$ (note that, in general, $\tilde{\mathcal{F}}'$ will not coincide with $\{\tilde{f}_i\}_{i\in I\backslash J}$, where $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$ is the standard dual frame of $\mathcal{F}$). Let $\tilde{T}_{I\backslash J}$ be the analysis operator for $\tilde{\mathcal{F}}'$. Then $T_{I\backslash J}^* \tilde{T}_{I\backslash J} = \mathbf{1}$, the identity operator on H. Define $L = \tilde{T}_{I\backslash J} T_J^*$. Then $T_J^* - T_{I\backslash J}^* L = 0$, so (14) is satisfied with $\gamma = 0$. □

Specializing to the case of removing a single element yields the following corollary, which will play an important role in the proof of Theorem 3 that is presented in Section 4 below.

Corollary 10 *Let* $\mathcal{F} = \{f_i\}_{i\in I}$ *be a frame for a Hilbert space* H*, with frame bounds* A*,* B*. Let* $j \in I$ *be given. If there exists a sequence* $a = (a_i)_{i\neq j} \in \ell^2(I\backslash\{j\})$ *such that*

$$\gamma \;=\; \Big\|f_j - \sum_{i\neq j} a_i f_i\Big\|^2 \;<\; \frac{A}{2}, \tag{18}$$

then $\mathcal{F}' = \{f_i\}_{i\neq j}$ *is a frame for* H *with frame bounds* $A' = \frac{A-2\gamma}{1+2\|a\|^2_{\ell^2}}$*,* $B' = B$*.*

PROOF. Set $J = \{j\}$, and define $L\colon \mathbf{C} \to \ell^2(I\backslash\{j\})$ by $L(c) = ca$. Then $\|L\| = \|a\|_{\ell^2}$, and

$$(T_J^* - T_{I\backslash J}^* L)(c) \;=\; cf_j - \sum_{i\neq j} ca_i f_i, \qquad c \in \mathbf{C}. \tag{19}$$

Hence

$$\gamma \;=\; \|T_J^* - T_{I\backslash J}^* L\|^2 \;=\; \Big\|f_j - \sum_{i\neq j} a_i f_i\Big\|^2, \tag{20}$$

so the result follows from Theorem 9. □

2.4 Modulation Spaces

The modulation spaces were introduced and extensively investigated by Feichtinger over the period 1980–1995, with some of the main references being (10; 11; 12; 13; 14; 15). The modulation space norms quantify the time-frequency content of a function or distribution, and appear naturally in mathematical problems involving time-frequency shifts. We refer to (17) for detailed discussion and applications.

For our purposes, the following special case of unweighted modulation spaces will be sufficient. Let $G(x) = 2^{d/4} e^{-\pi x\cdot x}$ be the Gaussian function, normalized so that $\|G\|_2 = 1$. Then for $1 \le p \le \infty$, the modulation space M^p consists of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$\begin{aligned} \|f\|_{M^p} \;&=\; \Big(\int_{\mathbf{R}^{2d}} |\langle f, \pi(z)G\rangle|^p \, dz\Big)^{1/p} \\ &=\; \Big(\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} |\langle f, M_\omega T_x G\rangle|^p \, dx\, d\omega\Big)^{1/p} \;<\; \infty, \end{aligned} \tag{21}$$

with the usual adjustment if $p = \infty$.

Remark 11 a. M^p is a Banach space for each $1 \le p \le \infty$. Any nonzero function $g \in M^1$ (including all Schwartz-class functions in particular) can be substituted for the Gaussian G in (21) to produce an equivalent norm for M^p, cf. Lemma 16 below.

b. $M^2 = L^2$, and $\mathcal{S} \subsetneq M^p \subsetneq M^q \subsetneq \mathcal{S}'$ for $1 \le p < q \le \infty$, where $\mathcal{S}$ is the Schwartz class.

c. If $1 \le p < \infty$ then $(M^p)' = M^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

d. M^p is isometric under time-frequency shifts, i.e.,

$$\forall\, z \in \mathbf{R}^{2d}, \quad \|\pi(z)f\|_{M^p} \;=\; \|f\|_{M^p}. \tag{22}$$

e. If Q is a cube in $\mathbf{R}^d$, then $h = \mathbf{1}_Q \in M^p$ for $1 < p \le \infty$.

The case $p = 1$ of the following proposition is a standard result for the modulation spaces. We will require the following extension to other values of p. A proof of this proposition was provided to us by K. Gröchenig and E. Cordero, and is reported in the Appendix.

Proposition 12 *Let $1 \le p, q, r \le \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, and let Λ be a sequence of points in $\mathbf{R}^{2d}$ satisfying $D^+(\Lambda) < \infty$. There there exists a constant $C = C(p, q, \Lambda) > 0$ such that*

$$\forall\, g \in M^p, \quad \forall\, f \in M^q, \quad \biggl(\sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda) g\rangle|^r\biggr)^{1/r} \;\le\; C\, \|g\|_{M^p}\, \|f\|_{M^q}. \tag{23}$$

2.5 A Recurrence Lemma

The following recurrence lemma will be used later in the proof of Theorem 3.

Lemma 13 *Let $\alpha_1, \dots, \alpha_r > 0$ be given, and fix $\delta > 0$. Then there exist infinitely many points $(n_j^1, \dots, n_j^r) \in \mathbf{Z}^d \times \cdots \times \mathbf{Z}^d$ such that for each $j = 1, 2, \dots$ we have*

$$|\alpha_1 n_j^1 - \alpha_k n_j^k| \;<\; \delta, \qquad k = 2, \dots, r. \tag{24}$$

PROOF. It suffices to prove the case $d = 1$. Let $\mathbf{T}^r = [0, \alpha_1) \times \cdots \times [0, \alpha_r)$ be the r-torus, and define a translation $T \colon \mathbf{T}^r \to \mathbf{T}^r$ by $T(x_1, \dots, x_r) = (x_1 +$

1 mod α_1, ..., $x_r + 1$ mod α_r). Let U be the open ball of radius $\delta/2$ centered at 0 in $\mathbf{T}^r$. Then, since T is a measure-preserving mapping, we have by the Poincare Recurrence Theorem ((24, p. 11) or (26, Thm. 1.4)) that almost every point of U returns to U infinitely often under iteration by T. Let $a \in U$ be any such point. Then there exist infinitely many positive integers $N_1 < N_2 < \cdots$ such that $T^{N_j}(a) \in U$, i.e., for each j we have

$$|(N_j + a) \bmod \alpha_k| \ < \ \frac{\delta}{2}, \qquad k = 1, \dots, r. \tag{25}$$

Hence, there exist integers n_j^k such that

$$|(N_j + a) - n_j^k \alpha_k| \ < \ \frac{\delta}{2}, \qquad k = 1, \dots, r, \tag{26}$$

and consequently,

$$|n_j^1 \alpha_1 - n_j^k \alpha_k| \ < \ \delta, \qquad k = 2, \dots, r. \tag{27}$$

By taking δ small enough, we are assured that the integers n_j^1 are distinct, which completes the proof. $\square$

3 Proof of Theorem 2

We will prove Theorem 2 in this section. We break the proof down into several smaller steps. We assume throughout this section that $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbf{R}^d)$ with frame bounds A, B, that g lies in M^p for some $1 \le p < 2$, and that $1 < D^+(\Lambda) < \infty$.

The frame operator for $\mathcal{G}(g, \Lambda)$ is $Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g$. The standard dual frame of $\mathcal{G}(g, \Lambda)$ is $\tilde{\mathcal{G}} = \{\tilde{g}_\lambda\}_{\lambda \in \Lambda}$ where $\tilde{g}_\lambda = S^{-1}(\pi(\lambda) g)$. If Λ is a lattice in $\mathbf{R}^{2d}$, then it can be shown that this dual frame is itself a Gabor frame of the form $\mathcal{G}(\gamma, \Lambda)$, but this need not be the case when Λ is not a lattice. For simplicity of notation, we will write

$$g_\lambda \ = \ \pi(\lambda) g, \tag{28}$$

but we emphasize that $\tilde{g}_\lambda$ need not be of the form $\pi(\lambda)\gamma$.

3.1 Goal

Our goal is to show that equation (5) holds, and that this implies that an infinite subset J of Λ can be found such that $\mathcal{G}(g, \Lambda\backslash J)$ is still a frame. To see how this second statement is a consequence of the first, recall from Theorem 6 that to show that there is an infinite subset of Λ which may be removed yet leave a frame, we need to show that there exists some (possibly different) infinite subset $J \subset \Lambda$ such that

$$\sup_{\lambda \in J} \langle g_\lambda, \tilde{g}_\lambda \rangle \;<\; 1. \tag{29}$$

Further, by Lemma 8, to do this it suffices to show that

$$\liminf_{r\to\infty} \inf_{z \in \mathbf{R}^{2d}} \frac{1}{|I(r,z)|} \sum_{\lambda \in I(r,z)} \langle g_\lambda, \tilde{g}_\lambda \rangle \;<\; 1, \tag{30}$$

where $I(r,z) = \Lambda \cap Q(r,z)$. We will show that the quantity on the left side of the preceding equation is actually bounded by $1/D^+(\Lambda)$. Hence, when $D^+(\Lambda) > 1$, there will be an infinite J for which (29) is satisfied.

For simplicity of notation, we will often use the abbreviation $I = I(r,z)$. Define the truncated frame operators

$$S_I f \;=\; \sum_{\lambda \in I} \langle f, g_\lambda \rangle\, g_\lambda \qquad \text{and} \qquad S_{\Lambda\backslash I} f \;=\; \sum_{\lambda \in \Lambda\backslash I} \langle f, g_\lambda \rangle\, g_\lambda. \tag{31}$$

We note the following basic facts.

Lemma 14 *a.* $\|g_\lambda\|_2^2 \le B$ *for each* $\lambda \in \Lambda$.

b. $\|\tilde{g}_\lambda\|_2^2 \le \frac{1}{A}$ *for each* $\lambda \in \Lambda$.

c. $\|S_I\|,\ \|S_{\Lambda\backslash I}\| \le \|S\| \le B$ *(operator norms).*

d. $\|S^{-1}\| \le \frac{1}{A}$.

e. The trace of $S_I S^{-1}$ *is* $\operatorname{tr}(S_I S^{-1}) \;=\; \sum_{\lambda \in I} \langle g_\lambda, \tilde{g}_\lambda \rangle$.

Thus, our goal is to show that

$$\liminf_{r\to\infty} \inf_{z \in \mathbf{R}^{2d}} \frac{1}{|I|} \operatorname{tr}(S_I S^{-1}) \;\le\; \frac{1}{D^+(\Lambda)}, \qquad I = I(r,z). \tag{32}$$

3.2 Some Notation

Let $\varepsilon > 0$ be fixed for the remainder of this proof. Since

$$D^+(\Lambda) \;=\; \limsup_{r\to\infty} \sup_{z\in\mathbf{R}^{2d}} \frac{|I(r,z)|}{r^{2d}}, \tag{33}$$

there exists a strictly increasing sequence $r_k \to \infty$ and points $z_k \in \mathbf{R}^{2d}$ such that

$$\forall\, k, \quad |I| \;=\; |I(r_k, z_k)| \;\geq\; r_k^{2d}\,(D^+(\Lambda) - \varepsilon). \tag{34}$$

3.3 One Gabor Orthonormal Basis and the HAP

Let $Q = Q(1,0) = [-\frac{1}{2}, \frac{1}{2}]^d$, and set $h = \mathbf{1}_Q$, the characteristic function of Q. Then

$$\mathcal{G}(h, \mathbf{Z}^{2d}) \;=\; \{\pi(\delta)h : \delta \in \mathbf{Z}^{2d}\} \;=\; \{h_\delta : \delta \in \mathbf{Z}^{2d}\} \tag{35}$$

is a Gabor orthonormal basis for $L^2(\mathbf{R}^d)$. We will use this system as a reference for comparison to $\mathcal{G}(g, \Lambda)$. We could use another Gabor orthonormal or Riesz basis, but by the Balian-Low theorem the generator of any such basis is limited in its joint time-frequency concentration. In particular, no generator of a Gabor Riesz basis can lie in M^1, cf. (15), (18). The particular Gabor system $\mathcal{G}(h, \mathbf{Z}^{2d})$ has the advantage that $h \in M^q$ for each $q > 1$.

We apply the Homogeneous Approximation Property for the Gabor orthonormal basis $\mathcal{G}(h, \mathbf{Z}^{2d})$ to the function g. In particular, by (5, Cor. 3.5), there exists an $R > 0$ such that

$$\forall\, z \in \mathbf{R}^{2d}, \quad \forall\, r \geq 0, \quad \forall\, \mu \in Q(r,z), \quad \|(\mathbf{1} - P_V)g_\mu\|_2 \;<\; \varepsilon, \tag{36}$$

where P_V is the orthogonal projection of $L^2(\mathbf{R}^d)$ onto

$$V \;=\; V(r+R, z) \;=\; \mathrm{span}\{h_\delta : \delta \in \mathbf{Z}^{2d} \cap Q(r+R, z)\}. \tag{37}$$

We will concentrate for a while on a specific $r = r_k > R$ and $z = z_k \in \mathbf{R}^{2d}$. We will suppress some indices and write

$$I \;=\; I(r_k, z_k) \;=\; \Lambda \cap Q(r_k, z_k), \tag{38}$$

$$\begin{aligned} V &= V(r_k+R, z_k) = \operatorname{span}\{h_\delta : \delta \in \mathbf{Z}^{2d} \cap Q(r_k+R, z_k)\}, && (39)\\ W &= V(r_k-R, z_k) = \operatorname{span}\{h_\delta : \delta \in \mathbf{Z}^{2d} \cap Q(r_k-R, z_k)\}, && (40)\\ U &= V(R, \lambda) = \operatorname{span}\{h_\delta : \delta \in \mathbf{Z}^{2d} \cap Q(R, \lambda)\}, && (41) \end{aligned}$$

and let P_V, P_W, P_U denote the orthogonal projection of $L^2(\mathbf{R}^d)$ onto V, W, and U, respectively. In this notation, note that the HAP (36) implies in particular that

$$\forall\, \lambda \in I, \quad \|(\mathbf{1} - P_V) g_\lambda\|_2 < \varepsilon, \tag{42}$$

and that

$$\forall\, \lambda \in \Lambda, \quad \|(\mathbf{1} - P_U) g_\lambda\|_2 < \varepsilon. \tag{43}$$

3.4 First Estimate

Recall that our goal is to estimate $\frac{1}{|I|}\operatorname{tr}(S_I S^{-1})$. Write

$$\operatorname{tr}(S_I S^{-1}) = \operatorname{tr}((\mathbf{1} - P_V) S_I S^{-1}) + \operatorname{tr}(P_V S_I S^{-1}). \tag{44}$$

For the first term on the right of (44), observe that

$$\begin{aligned} (\mathbf{1} - P_V) S_I S^{-1} f &= (\mathbf{1} - P_V)\Big(\sum_{\lambda \in I} \langle S^{-1} f, g_\lambda\rangle\, g_\lambda\Big) \\ &= \sum_{\lambda \in I} \langle f, \tilde{g}_\lambda\rangle\, (\mathbf{1} - P_V) g_\lambda. \end{aligned} \tag{45}$$

Computing the trace, applying the HAP in the form of (42), and using the boundedness of the norms of the dual frame elements, we have

$$\begin{aligned} \operatorname{tr}((\mathbf{1} - P_V) S_I S^{-1}) &= \sum_{\lambda \in I} \langle (\mathbf{1} - P_V) g_\lambda,\, \tilde{g}_\lambda\rangle \\ &\le \sum_{\lambda \in I} \|(\mathbf{1} - P_V) g_\lambda\|_2\, \|\tilde{g}_\lambda\|_2 \le \frac{\varepsilon |I|}{A^{1/2}}. \end{aligned} \tag{46}$$

3.5 Second Estimate

Now we will work on the second term on the right of (44). We will expand that term into three parts and then simplify by using the relations

$$P_V(\mathbf{1} - P_W) = P_{V \cap W^\perp}, \tag{47}$$

$$P_V P_W = P_W \qquad \text{(since } W \subset V\text{)}, \tag{48}$$

$$S = S_I + S_{\Lambda \setminus I}. \tag{49}$$

The three terms in the expansion are obtained as follows:

$$\begin{aligned}
\operatorname{tr}(P_V S_I S^{-1}) &= \operatorname{tr}(P_V (S - S_{\Lambda\setminus I}) S^{-1}) \\
&= \operatorname{tr}(P_V) - \operatorname{tr}(P_V(\mathbf{1} - P_W) S_{\Lambda\setminus I} S^{-1}) - \operatorname{tr}(P_V P_W S_{\Lambda\setminus I} S^{-1}) \\
&= \operatorname{tr}(P_V) - \operatorname{tr}(P_{V\cap W^\perp} S_{\Lambda\setminus I} S^{-1}) - \operatorname{tr}(P_W S_{\Lambda\setminus I} S^{-1}) \\
&\le \operatorname{tr}(P_V) + |\operatorname{tr}(P_{V\cap W^\perp} S_{\Lambda\setminus I} S^{-1})| + |\operatorname{tr}(P_W S_{\Lambda\setminus I} S^{-1})|. \qquad (50)
\end{aligned}$$

In the following we will bound each of these three terms separately.

3.5.1 First term

To estimate the first term in (50), note that the dimension of V is known because $\mathcal{G}(h, \mathbf{Z}^{2d})$ is an orthonormal basis. Consequently,

$$\operatorname{tr}(P_V) = \dim(V) = |\mathbf{Z}^{2d} \cap Q(r_k + R, z_k)| \le (r_k + R + 1)^{2d}. \tag{51}$$

3.5.2 Second term

For the second term in (50), note that since $\mathcal{G}(h, \mathbf{Z}^{2d})$ is an orthonormal basis, we have that

$$V \cap W^\perp = \operatorname{span}\{h_\delta : \delta \in \mathbf{Z}^{2d} \cap [Q(r_k + R, z_k) \setminus Q(r_k - R, z_k)]\}. \tag{52}$$

The set $Q(r_k + R, z_k) \setminus Q(r_k - R, z_k)$ is a “square annulus,” so

$$\begin{aligned}
\dim(V \cap W^\perp) &= |\mathbf{Z}^{2d} \cap [Q(r_k + R, z_k) \setminus Q(r_k - R, z_k)]| \\
&\le (r_k + R + 1)^{2d} - (r_k - R - 1)^{2d}. \qquad (53)
\end{aligned}$$

Now we apply the fact that

$$X = X^* \ge 0 \;\Rightarrow\; |\operatorname{tr}(XY)| \le \operatorname{tr}(X)\,\|Y\| \tag{54}$$

to compute that

$$|\operatorname{tr}(P_{V\cap W^\perp} S_{\Lambda\setminus I} S^{-1})| \le \operatorname{tr}(P_{V\cap W^\perp})\,\|S_{\Lambda\setminus I} S^{-1}\|$$

$$
\begin{aligned}
&\leq \dim(V \cap W^\perp)\, \|S_{\Lambda\setminus I}\|\, \|S^{-1}\| \\
&\leq \frac{B}{A}\,((r_k + R + 1)^{2d} - (r_k - R - 1)^{2d}). \qquad (55)
\end{aligned}
$$

3.5.3 Third term

Now we come to the third term in (50). Observe that

$$
P_W S_{\Lambda\setminus I} S^{-1} f \;=\; P_W \sum_{\lambda\in\Lambda\setminus I} \langle S^{-1}f, g_\lambda\rangle\, g_\lambda \;=\; \sum_{\lambda\in\Lambda\setminus I} \langle f, \tilde{g}_\lambda\rangle\, P_W g_\lambda. \qquad (56)
$$

Let

$$
D \;=\; \mathbf{Z}^{2d} \cap Q(r_k - R, z_k). \qquad (57)
$$

Then since $\mathcal{G}(h, \mathbf{Z}^{2d})$ is an orthonormal basis and since $W = \{h_\delta : \delta \in D\}$, we have $P_W h_\delta = h_\delta$ when $\delta \in D$ and $P_W h_\delta = 0$ otherwise. Hence

$$
\begin{aligned}
|\mathrm{tr}(P_W S_{\Lambda\setminus I} S^{-1})|^2 &= \Big| \sum_{\delta\in\mathbf{Z}^{2d}} \sum_{\lambda\in\Lambda\setminus I} \langle h_\delta, \tilde{g}_\lambda\rangle\, \langle P_W g_\lambda, h_\delta\rangle \Big|^2 \\
&= \Big| \sum_{\delta\in\mathbf{Z}^{2d}} \sum_{\lambda\in\Lambda\setminus I} \langle h_\delta, \tilde{g}_\lambda\rangle\, \langle g_\lambda, P_W h_\delta\rangle \Big|^2 \\
&= \Big| \sum_{\delta\in D} \sum_{\lambda\in\Lambda\setminus I} \langle h_\delta, \tilde{g}_\lambda\rangle\, \langle g_\lambda, h_\delta\rangle \Big|^2 \\
&\leq \Big(\sum_{\delta\in D} \sum_{\lambda\in\Lambda\setminus I} |\langle h_\delta, \tilde{g}_\lambda\rangle|^2 \Big) \Big(\sum_{\delta\in D} \sum_{\lambda\in\Lambda\setminus I} |\langle g_\lambda, h_\delta\rangle|^2 \Big). \qquad (58)
\end{aligned}
$$

We can bound the first factor on the right of (58) by using the fact that $\{\tilde{g}_\lambda\}_{\lambda\in\Lambda}$ is a frame for $L^2(\mathbf{R}^d)$ with frame bounds $\frac{1}{B}$, $\frac{1}{A}$:

$$
\sum_{\delta\in D} \sum_{\lambda\in\Lambda\setminus I} |\langle h_\delta, \tilde{g}_\lambda\rangle|^2 \;\leq\; \sum_{\delta\in D} \frac{1}{A}\, \|h_\delta\|_2^2 \;=\; \frac{|D|}{A} \;\leq\; \frac{(r_k - R + 1)^{2d}}{A}. \qquad (59)
$$

For the second factor on the right of (58), recall that $g \in M^p$ where $1 \leq p < 2$. Fix $p < s < 2$. Then

$$
\sum_{\delta\in D} \sum_{\lambda\in\Lambda\setminus I} |\langle g_\lambda, h_\delta\rangle|^2 \;\leq\; \Big(\sup_{\lambda\in\Lambda\setminus I,\, \delta\in D} |\langle g_\lambda, h_\delta\rangle|^{2-s} \Big) \sum_{\delta\in D} \sum_{\lambda\in\Lambda\setminus I} |\langle g_\lambda, h_\delta\rangle|^s. \qquad (60)
$$

We will estimate each of these two pieces separately.

To bound the first factor on the right of (60), consider a typical $\lambda \in \Lambda \backslash I$ and $\delta \in D$. We have that

$$|\langle g_\lambda, h_\delta \rangle| \;=\; |\langle g_\lambda, P_W h_\delta \rangle| \;\leq\; \|P_W g_\lambda\|_2 \, \|h_\delta\|_2 \;=\; \|P_W g_\lambda\|_2. \tag{61}$$

Letting $U = V(R, \lambda)$, we have by the HAP in the form of (43) that

$$\|(\mathbf{1} - P_U) g_\lambda\|_2 \;<\; \varepsilon. \tag{62}$$

However, $W = V(r_k - R, z_k)$ and $\lambda \notin I = I(r_k, z_k)$, so it follows that $U \subset W^\perp$ and $W \subset U^\perp$. Therefore $P_W \leq P_{U^\perp} = \mathbf{1} - P_U$, so

$$\forall\, \delta \in D, \quad \forall\, \lambda \in \Lambda \backslash I, \quad |\langle g_\lambda, h_\delta \rangle| \;\leq\; \|P_W g_\lambda\|_2 \;\leq\; \|(\mathbf{1} - P_U) g_\lambda\| \;<\; \varepsilon. \tag{63}$$

To estimate the second factor on the right of (60), let q be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{s}$. Note that $q > 1$, so $h_\delta \in M^q$. Since M^q is invariant under time-frequency shifts, we have $\|h_\delta\|_{M^q} = \|h\|_{M^q}$ for every δ. Since $g \in M^p$, we therefore have from Proposition 12 that

$$\begin{aligned} \sum_{\delta \in D} \sum_{\lambda \in \Lambda \backslash I} |\langle g_\lambda, h_\delta \rangle|^s &\leq \sum_{\delta \in D} C^s \, \|g\|_{M^p}^s \, \|h_\delta\|_{M^s}^s \\ &= C^s \, \|g\|_{M^p}^s \, \|h\|_{M^s}^s \, |D| \\ &\leq C_1 \, (r_k - R + 1)^{2d}. \end{aligned} \tag{64}$$

Combining the estimates (58)–(64) yields

$$|\mathrm{tr}(P_W S_{\Lambda \backslash I} S^{-1})|^2 \;\leq\; \frac{(r_k - R + 1)^{2d}}{A} \, \varepsilon^{2-s} \, C_1 \, (r_k - R + 1)^{2d}, \tag{65}$$

or

$$|\mathrm{tr}(P_W S_{\Lambda \backslash I} S^{-1})| \;\leq\; C_2 \, \varepsilon^{\frac{2-s}{2}} \, (r_k - R + 1)^{2d}. \tag{66}$$

3.6 Combine the Terms

Combining all the previous estimates yields for each $I = I(r_k, z_k)$ that

$$\frac{1}{|I|} \, \mathrm{tr}(S_I S^{-1}) \;\leq\; \frac{1}{|I|} \Big(|\mathrm{tr}((\mathbf{1} - P_V) S_I S^{-1})| \;+\; |\mathrm{tr}(P_V)|$$

$$
\begin{aligned}
&\quad + |\mathrm{tr}(P_{V\cap W^\perp} S_{\Lambda\setminus I} S^{-1})| \;+\; |\mathrm{tr}(P_W S_{\Lambda\setminus I} S^{-1}|\Big) \\
&\le \frac{\varepsilon}{A^{1/2}} \;+\; \frac{(r_k+R+1)^{2d}}{r_k^{2d}\,(D^+(\Lambda)-\varepsilon)} \\
&\quad + \frac{B\,((r_k+R+1)^{2d}-(r_k-R-1)^{2d})}{A\,r_k^{2d}\,(D^+(\Lambda)-\varepsilon)} \\
&\quad + \frac{C_2\,\varepsilon^{\frac{2-s}{2}}\,(r_k-R+1)^{2d}}{r_k^{2d}\,(D^+(\Lambda)-\varepsilon)},
\end{aligned} \tag{67}
$$

where we have bounded $1/|I|$ by using (34). Since z_k is one point in $\mathbf{R}^{2d}$ and since $r_k \to \infty$, we therefore have

$$
\liminf_{r\to\infty} \inf_{z\in\mathbf{R}^{2d}} \frac{1}{|I|}\,\mathrm{tr}(S_I S^{-1}) \;\le\; \frac{\varepsilon}{A^{1/2}} + \frac{1}{D^+(\Lambda)-\varepsilon} + 0 + \frac{C_2\,\varepsilon^{\frac{2-p}{2}}}{D^+(\Lambda)-\varepsilon}, \tag{68}
$$

where $I = I(r,z)$ in the equation above. Since ε was arbitrary, we conclude that

$$
\liminf_{r\to\infty} \inf_{z\in\mathbf{R}^{2d}} \frac{1}{|I|}\,\mathrm{tr}(S_I S^{-1}) \;\le\; \frac{1}{D^+(\Lambda)}, \tag{69}
$$

which was our goal. This completes the proof of Theorem 2.

4 Proof of Theorem 3

We will prove Theorem 3 in this section. We are given $g_k \in L^2(\mathbf{R}^d)$ and $\Lambda_k = \alpha_k\mathbf{Z}^d \times \beta_k\mathbf{Z}^d$ for $k = 1,\dots,r$, and we assume that $\mathcal{F} = \bigcup_{k=1}^r \mathcal{G}(g_k,\Lambda_k)$ is an overcomplete frame for its closed span H in $L^2(\mathbf{R})$. We must show that some infinite subset of some $\mathcal{G}(g_j,\Lambda_j)$ can be removed from $\mathcal{F}$ so that the remaining set is still a frame for H, and further this set can be chosen to consist of translates of a single generator g_i.

Let A, B denote the frame bounds for $\mathcal{F}$. For simplicity of notation, we will write the elements of $\mathcal{F}$ as

$$
\begin{aligned}
g^k_{m,n}(x) &= \pi(\beta_k m, \alpha_k n) g_k(x) \\
&= M_{\beta_k m} T_{\alpha_k n} g_k(x) \\
&= e^{2\pi i \beta_k m\cdot x} g_k(x-\alpha_k n),
\end{aligned} \tag{70}
$$

where $m, n \in \mathbf{Z}^d$ and $k = 1,\dots,r$.

Since $\mathcal{F}$ is overcomplete, there is some element which may be removed yet still leave a frame for H. Without loss of generality we may assume it is an element of $\mathcal{G}(g_1, \Lambda_1)$, say $h = g^1_{k,\ell}$. Since $h_{m,n} = e^{-2\pi i \beta_1 k \cdot \alpha_1 n} g^1_{m+k,n+\ell}$, the elements of $\mathcal{G}(h, \Lambda_1)$ are exactly the elements of $\mathcal{G}(g_1, \Lambda_1)$ except in a different order and multiplied by constants of magnitude 1 (here is one point where we make use of the assumption that the Λ_k are lattices). Without loss of generality we may therefore assume that the element removed is $g_1 = g^1_{0,0}$.

Define an index set

$$\Gamma_0 \;=\; (\{1, \dots, r\} \times \mathbf{Z}^d \times \mathbf{Z}^d) \setminus \{(1, 0, 0)\}. \tag{71}$$

Then

$$\mathcal{F}'_0 \;=\; \mathcal{F} \setminus \{g^1_{0,0}\} \;=\; \{g^k_{m,n}\}_{(k,m,n) \in \Gamma_0} \tag{72}$$

is a frame for H. We will show that there exist infinitely many indices $n_j \in \mathbf{Z}^d$ such that if we set

$$\Gamma_{n_j} \;=\; (\{1, \dots, r\} \times \mathbf{Z}^d \times \mathbf{Z}^d) \setminus \{(1, 0, n_j)\}, \tag{73}$$

then

$$\mathcal{F}'_{n_j} \;=\; \mathcal{F} \setminus \{g^1_{0,n_j}\} \;=\; \{g^k_{m,n}\}_{(k,m,n) \in \Gamma_{n_j}} \tag{74}$$

is also frame for h, and furthermore all of these frames $\mathcal{F}'_{n_j}$ have the same lower frame bound $L > 0$. It then follows from Theorem 6 that infinitely many elements may be removed from $\mathcal{F}$ yet leave a frame, and furthermore, this set to be removed is a subset of $\{g^1_{0,n_j}\}_{j=1}^{\infty}$, which is a set of translates of g_1.

Let S_0 be the frame operator for the frame $\mathcal{F}'_0$. Define

$$a^k_{m,n} \;=\; \langle S_0^{-1} g_1, g^k_{m,n} \rangle, \tag{75}$$

and set $a = (a^k_{m,n})_{(k,m,n) \in \Gamma_0}$. Note that $a \in \ell^2(\Gamma_0)$ since the scalars $a^k_{m,n}$ are the frame coefficients of $S_0^{-1} g$ with respect to the frame $\mathcal{F}'_0$.

Define

$$h_1 \;=\; \sum_{(m,n) \neq (0,0)} a^1_{m,n}\, g^1_{m,n} \tag{76}$$

and

$$h_k \;=\; \sum_{m,n\in\mathbf{Z}^d} a^k_{m,n}\, g^k_{m,n}, \qquad k=2,\dots,r. \tag{77}$$

Then

$$\begin{aligned}\sum_{k=1}^r h_k \;&=\; \sum_{(k,m,n)\in\Gamma_0} a^k_{m,n}\, g^k_{m,n}\\ &=\; \sum_{(k,m,n)\in\Gamma_0} \langle S_0^{-1} g_1, g^k_{m,n}\rangle\, g^k_{m,n}\\ &=\; S_0(S_0^{-1} g_1) \;=\; g_1. \end{aligned}\tag{78}$$

Fix $\varepsilon < \sqrt{A/2}$. Since translation is strongly continuous in $L^2(\mathbf{R}^d)$, there exists a $\delta > 0$ such that

$$|t| < \delta \quad\Rightarrow\quad \|h_k - T_t h_k\|_2 \;<\; \frac{\varepsilon}{r-1}, \quad k = 2,\dots,r. \tag{79}$$

By Lemma 13, there exist points $(n^1_j,\dots,n^r_j) \in \mathbf{Z}^d \times \cdots \times \mathbf{Z}^d$ for $j = 1,2,\dots$ such that

$$|\alpha_1 n^1_j - \alpha_k n^k_j| \;<\; \delta, \qquad k=2,\dots,r. \tag{80}$$

We will show that for each j,

$$\gamma_j \;=\; \Bigl\| g^1_{0,n^1_j} - \sum_{(k,m,n)\in\Gamma_{n^1_j}} a^k_{m,n-n^k_j}\, g^k_{m,n} \Bigr\|_2^2 \;\le\; \varepsilon^2 \;<\; \frac{A}{2}. \tag{81}$$

Consequently, by Corollary 10 we will have that $\mathcal{F}'_{n^1_j}$ is a frame for H with lower frame bound $A'_j = \frac{A-2\gamma_j}{1+2\|a\|^2_{\ell^2}}$. Since $A'_j \ge \frac{A-2\varepsilon^2}{1+2\|a\|^2_{\ell^2}} > 0$, all the frames $\mathcal{F}'_{n^1_j}$ will share the same single positive lower frame bound, and the proof will be complete.

To prove (81), we first use (79) and (80) to compute that

$$\Bigl\|\sum_{k=2}^r (T_{\alpha_k n^k_j} h_k - T_{\alpha_1 n^1_j} h_k)\Bigr\|_2 \;\le\; \sum_{k=2}^r \|h_k - T_{\alpha_1 n^1_j - \alpha_k n^k_j} h_k\|_2 \;<\; \varepsilon. \tag{82}$$

Therefore, since $g^1_{0,n^1_j} = T_{\alpha_1 n^1_j} g_1$,

$$\begin{aligned}
&\Big\| g^1_{0,n^1_j} - \sum_{k=1}^r T_{\alpha_k n^k_j} h_k \Big\|_2 \\
&\le \Big\| T_{\alpha_1 n^1_j} g_1 - \sum_{k=1}^r T_{\alpha_1 n^1_j} h_k \Big\|_2 + \Big\| \sum_{k=2}^r (T_{\alpha_1 n^1_j} h_k - T_{\alpha_k n^k_j} h_k) \Big\|_2 \\
&< \ 0 + \varepsilon \ = \ \varepsilon.
\end{aligned} \tag{83}$$

Finally,

$$\sum_{k=1}^r T_{\alpha_k n^k_j} h_k \ = \sum_{(k,m,n)\in\Gamma_0} a^k_{m,n}\, g^k_{m,n+n^k_j} \ = \sum_{(k,m,n)\in\Gamma_{n^1_j}} a^k_{m,n-n^k_j}\, g^k_{m,n}, \tag{84}$$

so the proof is complete.

A Appendix: Proof of Proposition 12

We will prove Proposition 12 in this section. We thank Karlheinz Gröchenig and Elena Cordero for providing this proof, which is related to techniques developed in (6).

We obtain this proof by applying interpolation. We refer to (3) for background on interpolation in general, and to (10; 12; 14) for results on the interpolation properties of the modulation and Wiener amalgam spaces.

Definition 15 *The* Short-Time Fourier Transform *of a tempered distribution f with respect to a window function g is*

$$V_g f(z) \ = \ \langle f, \pi(z) g\rangle, \qquad z \in \mathbf{R}^{2d}, \tag{A.1}$$

whenever this is defined.

In particular, letting $G(x) = 2^{d/4} e^{-\pi x\cdot x}$ denote the Gaussian window (normalized so that $\|G\|_2 = 1$), we can restate the definition of the modulation space M^p given in Section 2.4 as follows:

$$M^p \ = \ \{f \in \mathcal{S}'(\mathbf{R}^d) : \|f\|_{M^p} = \|V_G f\|_{L^p} < \infty\}. \tag{A.2}$$

The following lemma deals with the effect of replacing the Gaussian G by another function g. An extension of this lemma shows that $\|V_g f\|_{L^p}$ is actually an equivalent norm for M^p for each nonzero function $g \in M^1$.

Lemma 16 *If $1 \leq p \leq \infty$, then*

$$\forall\, g \in M^1, \quad \forall\, f \in M^p, \quad \|V_g f\|_{L^p} \;\leq\; \|g\|_{M^1}\, \|f\|_{M^p}. \tag{A.3}$$

PROOF. From (17, eq. (11.31)), there exists a constant C such that

$$\|V_g f\|_{L^p} \;\leq\; C\, \|V_g G\|_{L^1}\, \|V_G f\|_{L^p} \;=\; C\, \|g\|_{M^1}\, \|f\|_{M^p}. \tag{A.4}$$

Further, the value of C is determined by the weight used to define the modulation space. Since we are dealing only with the unweighted case (or, equivalently, the weight is identically 1), it can be shown that the constant is $C = 1$. □

Let $\mathcal{C} = \mathcal{C}(\mathbf{R}^d)$ denote the space of all bounded, continuous functions on $\mathbf{R}^d$ under the L^∞ norm.

Let $Q = [-\frac{1}{2}, \frac{1}{2}]^d$. Define the Wiener amalgam space $W(\mathcal{C}, \ell^p)$ to be the space of all continuous functions f on $\mathbf{R}^d$ for which the norm

$$\|f\|_{W(\mathcal{C},\ell^p)} \;=\; \left(\sum_{k \in \mathbf{Z}^d} \|f \cdot \mathbf{1}_{Q+k}\|_\infty^p \right)^{1/p} \tag{A.5}$$

is finite, with the usual adjustment if $p = \infty$. Note that $W(\mathcal{C}, \ell^\infty) = L^\infty \cap \mathcal{C} = \mathcal{C}$, while $W(\mathcal{C}, \ell^p)$ is a proper subset of $L^p \cap \mathcal{C}$ when $p < \infty$. We refer to (17; 19; 20) for background information on Wiener amalgam spaces,

For the convenience of the reader we recall the main result of interpolation theory (see Chapters 2 and 4 in (3) for precise definitions). For two Banach function spaces S_1, S_2 that are subspaces of a Hausdorff topological vector space U, we denote by $[S_1, S_2]_\theta$ the interpolation space obtained by the complex interpolation method with parameter $\theta \in [0, 1]$. In particular, as shown in (10; 12; 14), the interpolation spaces for the Wiener amalgam spaces and the modulation spaces are given by:

$$[W(\mathcal{C}, \ell^p),\, W(\mathcal{C}, \ell^q)]_\theta \;=\; W(\mathcal{C}, \ell^r), \tag{A.6}$$

and

$$[M^p,\, M^q]_\theta \;=\; M^r, \tag{A.7}$$

where

$$\frac{1}{r} \;=\; \frac{1-\theta}{p} + \frac{\theta}{q}. \tag{A.8}$$

Furthermore, if S_1, S_2 are subspaces of U and R_1, R_2 are subspaces of V, and if $T\colon U \to V$ is a linear operator such that its restrictions $T_1\colon S_1 \to R_1$ and $T_2\colon S_2 \to R_2$ are bounded operators, then its restriction

$$T_\theta\colon [S_1, S_2]_\theta \to [R_1, R_2]_\theta \tag{A.9}$$

is also a bounded operator, with norm

$$\|T_\theta\|_{\mathcal{B}([S_1,S_2]_\theta,[R_1,R_2]_\theta)} \;\le\; \|T_1\|^{1-\theta}_{\mathcal{B}(S_1,R_1)}\, \|T_2\|^{\theta}_{\mathcal{B}(S_2,R_2)}, \tag{A.10}$$

where $\mathcal{B}(S, R)$ denotes the Banach space of all bounded operators from S to R.

The following proposition is the key step in the proof of Proposition 12.

Proposition 17 *Let $1 \le p, q, r \le \infty$ be such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$. If $g \in M^p$ and $f \in M^q$, then $V_g f \in W(\mathcal{C}, \ell^r)$. Further, there exists a constant $C = C(p, q)$ such that*

$$\forall\, g \in M^p, \quad \forall\, f \in M^q, \quad \|V_g f\|_{W(\mathcal{C},\ell^r)} \;\le\; C\, \|g\|_{M^p}\, \|f\|_{M^q}. \tag{A.11}$$

PROOF. First let us show that $V_g f$ is bounded and continuous. Since $\frac{1}{p}+\frac{1}{q} \ge 1$, it follows that $p \le q'$. Hence $g \in M^p \subset M^{q'}$ while $f \in M^q$. By Remark 11, if $1 \le q < \infty$ then $M^{q'}$ is the dual of M^q, while if $1 < q \le \infty$ then M^q is the dual of $M^{q'}$. In either case, we can conclude that

$$|V_g f(z)| \;=\; |\langle f, \pi(z)g\rangle| \;\le\; \|f\|_{M^q}\, \|\pi(z)g\|_{M^{q'}} \;\le\; \|f\|_{M^q}\, \|g\|_{M^p}. \tag{A.12}$$

In particular, $V_g f$ is bounded. Now, at least one of p or q is finite, and since $|V_g f(\omega, t)| = |V_f g(-\omega, -t)|$, it suffices to consider $p < \infty$. Letting G denote the Gaussian function, we have for z, $z' \in \mathbf{R}^{2d}$ that

$$\begin{aligned} |V_g f(z) - V_g f(z')| \;&=\; |\langle f, \pi(z)g - \pi(z')g\rangle| \\ &\le\; \|f\|_{M^q}\, \|\pi(z)g - \pi(z')g\|_{M^p}. \\ &=\; \|f\|_{M^q}\, \|V_G(\pi(z)g) - V_G(\pi(z')g)\|_{L^p}. \end{aligned} \tag{A.13}$$

Now write $z = (x, \omega)$ and $z' = (x', \omega')$, and define

$$\phi_{z,z'}(y, \xi) = e^{2\pi i[(\omega'-\xi)\cdot x' - (\omega-\xi)\cdot x)]}. \tag{A.14}$$

Then

$$|V_G(\pi(z)g) - V_G(\pi(z')g)| \;=\; |T_z V_G g - \phi_{z,z'}\, T_{z'} V_G|, \tag{A.15}$$

where T_z is the translation operator on $\mathbf{R}^{2d}$, so

$$\begin{aligned} &\|V_G(\pi(z)g) - V_G(\pi(z')g)\|_{L^p} \\ &\;\le\; \|T_z V_G g - T_{z'} V_G g\|_{L^p} \;+\; \|T_{z'} V_G g - \phi_{z,z'}\, T_{z'} V_G\|_{L^p} \\ &\;\to 0 \quad \text{as } z \to z', \end{aligned} \tag{A.16}$$

the convergence in the first term following from the fact that translation is strongly continuous on L^p when $p < \infty$, and in the second term from the Lebesgue Dominated Convergence Theorem. Consequently $V_g f$ is continuous on $\mathbf{R}^{2d}$.

To complete the proof, we now progress through several cases.

Case 1: $p = 1$, $1 \le q \le \infty$, $r = q$.

If $q = 1$ then we have $f,\, g \in M^1$, so $V_g f \in W(\mathcal{C}, \ell^1)$ by (17, Prop. 12.1.11). Further, by (17), there exists a constant $C > 0$ such that

$$\|V_g f\|_{W(\mathcal{C},\ell^1)} \;\le\; C\, \|g\|_{M^1}\, \|f\|_{M^1}. \tag{A.17}$$

In fact, the proof of this result shows that it suffices to take $C = \|\mathbf{1}_{[-2,1]^d}\|_{L^1} = 3^d$. On the other hand, if $q = \infty$ then by Lemma 16,

$$\|V_g f\|_{W(\mathcal{C},\ell^\infty)} \;=\; \|V_g f\|_{L^\infty} \;\le\; \|g\|_{M^1}\, \|f\|_{M^\infty}. \tag{A.18}$$

The result for $1 < q < \infty$ then follows from interpolating between these two endpoints. In fact, if we set $\theta = 1/q$ then

$$\frac{1}{q} \;=\; \frac{\theta}{1} + \frac{1-\theta}{\infty}, \tag{A.19}$$

and we have

$$[W(\mathcal{C}, \ell^1),\, W(\mathcal{C}, \ell^\infty)]_\theta = W(\mathcal{C}, \ell^q) \quad \text{and} \quad [M^1,\, M^\infty]_\theta \;=\; M^q. \tag{A.20}$$

Hence, with g fixed, applying interpolation to the mapping $f \mapsto V_g f$ therefore yields

$$\|V_g f\|_{W(\mathcal{C},\ell^q)} \le (3^d \|g\|_{M^1})^\theta (\|g\|_{M^1})^{1-\theta} \|f\|_{M^q}$$
$$= 3^{d/q} \|g\|_{M^1} \|f\|_{M^q}. \tag{A.21}$$

Case 2: $\frac{1}{p} + \frac{1}{q} = 1$, $r = \infty$.

In this case we have $p = q'$. We have already shown that $V_g f \in L^\infty \cap \mathcal{C} = W(\mathcal{C}, \ell^\infty)$, and by (A.12)

$$\|V_g f\|_{W(\mathcal{C},\ell^\infty)} = \|V_g f\|_{L^\infty} \le \|g\|_{M^{q'}} \|f\|_{M^q}. \tag{A.22}$$

Case 3: General case, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.

By Case 1, we have

$$\|V_g f\|_{W(\mathcal{C},\ell^q)} \le (3^{d/q} \|f\|_{M^q}) \|g\|_{M^1}, \tag{A.23}$$

and by Case 2, we have

$$\|V_g f\|_{W(\mathcal{C},\ell^\infty)} \le \|f\|_{M^q} \|g\|_{M^{q'}}. \tag{A.24}$$

Set $\theta = q/r$. Then $0 \le \theta \le 1$ and

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{\infty} \quad \text{and} \quad \frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{q'}. \tag{A.25}$$

Further,

$$[W(\mathcal{C},\ell^q), W(\mathcal{C},\ell^\infty)]_\theta = W(\mathcal{C},\ell^r) \quad \text{and} \quad [M^1, M^{q'}]_\theta = M^p. \tag{A.26}$$

With f fixed, applying interpolation to the mapping $g \mapsto V_g f$ therefore yields

$$\|V_g f\|_{W(\mathcal{C},\ell^r)} \le (3^{d/q} \|f\|_{M^q})^\theta (\|f\|_{M^q})^{1-\theta} \|g\|_{M^p}$$
$$= 3^{d/r} \|f\|_{M^q} \|g\|_{M^p}, \tag{A.27}$$

which completes the proof. □

The significance of membership of $V_g f$ in the Wiener amalgam space is made clear by the next result, which follows from (17, Prop. 11.1.4) and states that the ℓ^r sequence-space norm of a set of separated samples of a function in

$W(\mathcal{C}, \ell^r)$ is bounded by the amalgam space norm of that function. A sequence Λ is *separated* if there exists some $\delta > 0$ such that any two different points of Λ are at least a distance δ apart.

Proposition 18 *Let $1 \le r \le \infty$ be given. If Λ is separated, then there exists a constant $C = C(r, \Lambda) > 0$ such that*

$$\forall\, F \in W(\mathcal{C}, \ell^r), \quad \Big(\sum_{\lambda\in\Lambda} |F(\lambda)|^r\Big)^{1/r} \;\le\; C\, \|F\|_{W(\mathcal{C},\ell^r)}. \tag{A.28}$$

A proof of Proposition 12 now follows from combining Lemma 17 with Proposition 18.

Proof of Proposition 12 Assume $D^+(\Lambda) < \infty$. Then by (5, Lemma 2.3), Λ can be written as a finite union of disjoint sequences $\Lambda_1, \dots, \Lambda_k$ each of which is separated. If $g \in M^p$ and $f \in M^q$, then $V_g f \in W(\mathcal{C}, \ell^r)$ by Lemma 17. Hence, by Proposition 18, we have for each $j = 1, \dots, k$ that

$$\begin{aligned} \sum_{\lambda\in\Lambda_j} |\langle f, \pi(\lambda) g\rangle|^r &= \sum_{\lambda\in\Lambda_j} |V_g f(\lambda)|^r \\ &\le C_1\, \|V_g f\|^r_{W(\mathcal{C},\ell^r)} \\ &\le C_2\, \|g\|^r_{M^p}\, \|f\|^r_{M^q}. \end{aligned} \tag{A.29}$$

Summing over j then completes the proof. □

References

[1] R. Balan, P. G. Casazza, C. Heil, and Z. Landau, Deficits and excesses of frames, Adv. Comput. Math., Special Issue on Frames, 2002, to appear.

[2] R. Balan, I. Daubechies, V. Vaishampayan, The analysis and design of windowed Fourier frame based multiple description encoding schemes, IEEE Trans. Inform. Theory, 46 (2000) 2491–2536.

[3] J. Bergh and J. Löfström, Interpolation Spaces, An Introduction, Springer–Verlag, New York, 1976.

[4] P. Casazza and J. Kovačević, Equal-norm tight frames with erasures, Adv. Comput. Math., Special Issue on Frames, 2002, to appear.

[5] O. Christensen, B. Deng, and C. Heil, Density of Gabor frames, Appl. Comput. Harmon. Anal., 7 (1999) 292–304.

[6] E. Cordero and K. Gröchenig, Time-frequency analysis of localization operators, preprint (2002).

[7] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.

[8] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc., 72 (1952) 341–366.

[9] Y. C. Eldar and H. Bölcskei, Geometrically uniform frames, IEEE Trans. Inform. Theory, to appear.

[10] H. G. Feichtinger, Banach convolution algebras of Wiener type, in: Functions, Series, Operators, Proc. Conf. Budapest, Vol. 38, Colloq. Math. Soc. János Bolyai, 1980, pp. 509–524.

[11] H. G. Feichtinger, On a new Segal algebra, Monatsh. Math., 92 (1981) 269–289.

[12] H. G. Feichtinger, Banach spaces of distributions of Wiener's type and interpolation, in: Functional Analysis and Approximation (Oberwolfach, 1980), Internat. Ser. Numer. Math., Vol. 60, Birkhäuser, Basel, 1981, pp. 153–165.

[13] H. G. Feichtinger and K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions, I, J. Funct. Anal., 86 (1989) 307–340.

[14] H. G. Feichtinger and K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions, II, Monatsh. Math., 108 (1989) 129–148.

[15] H. G. Feichtinger and K. Gröchenig, Gabor frames and time-frequency analysis of distributions, J. Funct. Anal., 146 (1997) 464–495.

[16] V. K. Goyal, J. Kovačević, and J. A. Kelner, Quantized frames with erasures, Appl. Comput. Harmon. Anal., 10 (2001) 203–233.

[17] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2001.

[18] K. Gröchenig, D. Han, C. Heil, and G. Kutyniok, The Balian–Low Theorem for symplectic lattices in higher dimensions, Appl. Comput. Harmon. Anal., 13 (2002) 169–176.

[19] C. Heil, An introduction to weighted Wiener amalgams, in: R. Ramakrishnan and S. Thangavelu (Eds.), Proc. International Conference on Wavelets and their Applications (Chennai, January 2002), Allied Publishers, New Delhi, to appear.

[20] C. E. Heil and D. F. Walnut, Continuous and discrete wavelet transforms, SIAM Review, 31 (1989) 628–666.

[21] A. J. E. M. Janssen, A density theorem for time-continuous filter banks, in: Y. Y. Zeevi and R. R. Coifman (Eds.), Signal and image representation in combined spaces, Wavelet Anal. Appl., Vol. 7, Academic Press, San Diego, CA, 1998, pp. 513–523.

[22] Y. Liu and Y. Wang, The uniformity of non-uniform Gabor bases, Adv. Comput. Math., Special Issue on Frames, 2002, to appear.

[23] J. Ramanathan and T. Steger, Incompleteness of sparse coherent states, Appl. Comput. Harmon. Anal., 2 (1995) 148–153.

[24] Ya. G. Sinaĭ, ed., Dynamical systems, II. Ergodic theory with applications to dynamical systems and statistical mechanics (translated from the Russian), Encyclopaedia of Mathematical Sciences, 2, Springer–Verlag,

Berlin, 1989.
[25] T. Strohmer and S. Beaver, Optimal OFDM system design for time-frequency dispersive channels, IEEE Trans. Communications, to appear.
[26] P. Walters, An Introduction to Ergodic Theory, Springer–Verlag, New York, 1982.