Horseshoes And Nonintegrability In The Restricted Case Of A Spinless Axisymmetric Rigid Body In A Central Gravitational Field

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Abstract.
The purpose of this paper is to study the motion of a spinless axisymmentric rigid body in a Newtonian field when we suppose the motion of the center of mass of the rigid body is on a Keplerian orbit. In this case the system can be reduced to a Hamiltonian system with configuration space a two-dimensional sphere. We prove that the restricted planar motion is analytical nonintegrable and we find horseshoes due to the eccentricity of the orbit. In the case $I_3/I_1 > 4/3$, we prove that the system on the sphere is also analytical nonintegrable.

Key words: horseshoes, analytic integrability, rigid body problem

1. Introduction

The purpose of this paper is to study the integrability of a Hamiltonian dynamical system associated to the motion of a rigid body in a central gravitational field. We prove that the spinless axisymmetric rigid body, which is completely integrable in an uniform field (the Lagrange case), is analytical nonintegrable in a central gravitational field and a chaotic motion of the internal rotation occurs.

In the restricted three-body problem there have been published many papers, starting with Poincaré 1890 (Poincaré, 1899) to the more recently textbooks such as (Szebehely, 1967) or (Marchal, 1990). On the other hand, the chaotic motion of a rigid body (namely the existence of horseshoes and Arnold diffusion) has been studied for some mechanical systems as in (Holmes *et al.*, 1983) or (Gray *et al.*, 1992).

The rigid body problem in celestial mechanics appeared with the paper by Duboshin in 1958 ((Duboshin, 1958)). Meantime there have been published many papers following two directions: in one direction the complete interaction between the motion of the centers of mass and the attitude motion (i.e. the motion around the center of mass) has been considered and the studies have concerned, primary, the existence and stability analysis of special solutions (see for instance (Eremenko, 1983), (Cid *et al.*, 1985) or (Wang *et al.*, 1992)); in another direction, the motion

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of the CM has been decoupled from the attitude motion and usually just the first correction in the attitude motion has been kept (see for instance the study of (Beletskii, 1966) or the papers of (Teofilatto *et al.*, 1992) or (Celletti *et al.*, 1992)).

Our study follows the second direction and the CM is supposed to move on an unperturbed Keplerian orbit. We also suppose to have an axisymmetric rigid body without spinning. This will simplify enough the equation of motions so that we will be able to handle these equations to prove the chaotic behaviour of the solution.

2. The Statement of the Problem

Suppose we have a rigid body m in a central gravitational field (i.e. of the form $-1/r$) whose center of attraction (O) is supposed to be a point-like (or homogeneous spherical) mass M.

To describe the rigid body we use two coordinate systems whose origins are at the center of attraction: one fixed called the fixed system (ξ, η, ζ) and another corresponding to the principal axes of the body (i.e. in which the tensor of internal momenta diagonalizes), called the body system with coordinates (x, y, z) - see fig.1. The transition from a coordinate system to another is given by a 3×3 matrix from $SO(3)$ denoted A so that $x^2 + y^2 + z^2 = \xi^2 + \eta^2 + \zeta^2 = r^2$. Thus the configuration space of the body is $K = \mathbf{R}^3 \times SO(3)$. We denote by G the constant of attraction.

The potential energy is given by the MacCallagh's formula (see (Goldstein, 1980)) and the kinetic energy is obtained via Köenig's theorem. By adding these two energies we get the Hamiltonian H, which is a function defined on the cotangent boundle to the manifold K, $H : T^*K \to \mathbf{R}$ given by:

$$\begin{aligned} H = \tfrac{p_\xi^2+p_\eta^2+p_\zeta^2}{2m} + \tfrac{l_1^2}{2I_1} + \tfrac{l_2^2}{2I_2} + \tfrac{l_3^2}{2I_3} & -\tfrac{GMm}{r} + \tfrac{GM}{2r^3}[3(\tfrac{x^2}{r^2}I_1 + \tfrac{y^2}{r^2}I_2 + \tfrac{z^2}{r^2}I_3) - \\ & -(I_1 + I_2 + I_3)] + \mathcal{O}(\tfrac{1}{r^4}) \end{aligned} \tag{1}$$

The phase space will be T^*K parametrized by $(\xi, \eta, \zeta, A; p_\xi, p_\eta, p_\zeta, l_1, l_2, l_3)$ (see (Holmes *et al.*, 1983)) where (p_ξ, p_η, p_ζ) are the components in the fixed system of the CM linear momentum ($p_\xi = m\dot{\xi}$, $p_\eta = m\dot{\eta}$, $p_\zeta = m\dot{\zeta}$) and (l_1, l_2, l_3) are the components in the body system of the internal angular momentum (i.e. the rigid body angular momentum with respect to the CM).

Now we make our assumptions:

A1. The attitude motion does not influence the motion of the CM;

A2. The higher order terms in the Hamiltonian are negligible;

A3. The body is axisymmetric: $I_1 = I_2 \neq I_3$;

A4. The body is without spinning: $l_3 = 0$

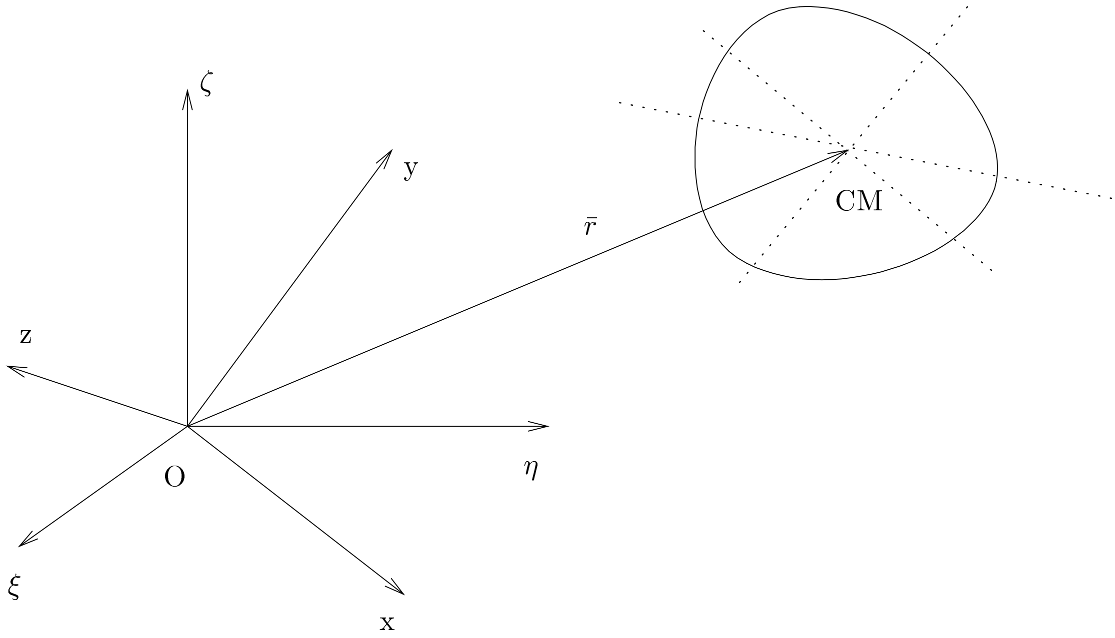

Fig. 1. The fixed and rigid body frames.

The first assumption says that the state space of the attitude motion is $SO(3)$ and the motion of the CM is given by an unperturbed Keplerian orbit that can be parametrized be:

$$\begin{aligned} \xi &= r\,cos\,v \\ \eta &= r\,sin\,v \\ \zeta &= 0 \end{aligned} \tag{2}$$

and:

$$\frac{1}{r} = \frac{1}{p}(1 + \varepsilon cos\,v) \tag{3}$$

$$r^2\dot{v} = C \tag{4}$$

$$GMp = C^2 \tag{5}$$

where: v is the true anomaly (is the planar angle measured between the position vector and the apocenter vector), p is the parameter of the orbit, ε the eccentricity and C the constant of areas. We shall consider only the cases $0 \leq \varepsilon < 1$, namely the circular and elliptic orbits. Then, from (4) and (5) we see that we can invert the dependency $v = v(t)$ into $t = t(v)$ and:

$$\frac{d}{dt} = \frac{C}{r^2}\frac{d}{dv} \tag{6}$$

The second assumption tells us that we can neglect the term $\mathcal{O}(\frac{1}{r^4})$ in the Hamiltonian. Even more, since the terms containing only r give no contribution to the equation of the attitude motion, we can cancel them from the Hamiltonian. Thus, after the first two assumptions, the equivalent Hamiltonian for the attitude motion is given by:

$$H_2(A; l_1, l_2, l_3; t) = \frac{l_1^2}{2I_1} + \frac{l_2^2}{2I_2} + \frac{l_3^2}{2I_3} + \frac{3GM}{2r^3}(\frac{x^2}{r^2}I_1 + \frac{y^2}{r^2}I_2 + \frac{z^2}{r^2}I_3) \tag{7}$$

where (x, y, z) are connected with (ξ, η, ζ) via $A(t)$.

Using the third assumption, the Hamiltonian turns into:

$$H_2 = \frac{l_1^2 + l_2^2}{2I_1} + \frac{l_3^2}{2I_3} - \frac{3GM}{2r^5}(I_1 - I_3)z^2 + \frac{3GM}{2r^3}I_1 \tag{8}$$

Since it is invariant under the rotation around the z-axis, it follows that l_3 is a constant of motion. Our study concerns the case $l_3 = 0$, i.e. the spinless rigid body. Because of this symmetry we can reduce the system to the quotient space $SO(3)/SO(2) \simeq S^2$ and we get a Hamiltonian system on T^*S^2 with configuration space a 2-dimensional sphere. To do this, we use the Euler's angles (Φ, θ, Ψ) and the Euler parametrization of $SO(3)$ - see fig.2. Then, the reduced Hamiltonian defined on $T^*S^2 \times \mathbf{R}$ takes the form:

$$H_{red}(\Phi, \theta; p_\Phi, p_\theta; t) = \frac{{p_\Phi}^2}{2I_1 \sin^2\theta} + \frac{{p_\theta}^2}{2I_1} - \frac{3GM}{2r^3}(I_1 - I_3)\sin^2\theta \frac{1 - \cos 2(\Phi - v)}{2} \tag{9}$$

and the problem is to study the (analytic) integrability of this Hamiltonian system.

We mention that the system has two singular points on the sphere, namely $\theta = 0$ and $\theta = \pi$ (the north and south poles), and this is due to the Euler's parametrization of the $SO(3)$.

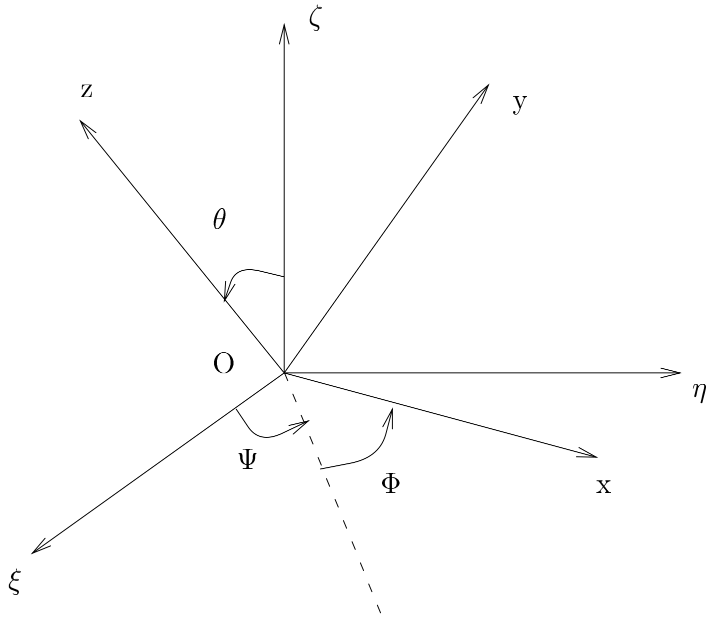

Fig. 2. The Euler's angles.

3. The Analysis of the Hamiltonian System on the Sphere

As we have seen before, our problem can be reduced on $T^*S^2 \times \mathbf{R}$ with the Hamiltonian (9). The canonical equations can be written now as:

$$\begin{array}{l} \dot{\Phi} = \frac{\partial H}{\partial p_\Phi} = \frac{p_\Phi}{I_1 \sin^2\theta} \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I_1} \\ \dot{p_\Phi} = -\frac{\partial H}{\partial \Phi} = \frac{3GM}{2r^3}(I_1 - I_3)\sin^2\theta \sin 2(\Phi - v) \\ \dot{p_\theta} = -\frac{\partial H}{\partial \theta} = \frac{{p_\Phi}^2 \cos\theta}{I_1 \sin^3\theta} + \frac{3GM}{2r^3}(I_1 - I_3)\sin 2\theta \frac{1-\cos 2(\Phi - v)}{2} \end{array} \qquad (10)$$

Now we prefer to change the time variable to v using (6) and (3). Even more, we make a change of variable: $\varphi = 2(\Phi - v)$. Then (10) is brought into the following form:

$$\begin{array}{l} \frac{d\varphi}{dv} = \frac{2r^2}{CI_1}\frac{p_\Phi}{sin^2\theta} - 2 \\ \frac{d\theta}{dv} = \frac{r^2}{CI_1}p_\theta \\ \frac{dp_\Phi}{dv} = \frac{3GM}{2rC}(I_1 - I_3)sin^2\theta \sin\varphi \\ \frac{dp_\theta}{dv} = \frac{r^2}{CI_1}\frac{{p_\Phi}^2\cos\theta}{sin^3\theta} + \frac{3GM}{2rC}(I_1 - I_3)\sin 2\theta \frac{1-\cos\varphi}{2} \end{array} \qquad (11)$$

There are two zeros of the vector field given by:

$$P_1 = (0, \frac{\pi}{2}, \frac{CI_1}{r^2}, 0) \quad , \quad P_2 = (\pi, \frac{\pi}{2}, \frac{CI_1}{r^2}, 0) \tag{12}$$

where $P = (\varphi, \theta, p_\Phi, p_\theta)$ is the parametrization of T^*S^2.

On the general case, when the CM is moving on an elliptic orbit, we have the following result:

LEMMA 1. *The following manifold:*

$$M_{inv} = \{\theta = \frac{\pi}{2}, p_\theta = 0\} \subset T^*S^2 \tag{13}$$

is an invariant manifold for (11). *Even more,* M_{inv} *is diffeomorphic equivalent with* T^*S^1. □

This fact comes from a very simple checking of the second and fourth equations of (11). On the other hand, the above manifold is invariant in both circular and elliptic motions of the CM so that this is an important information about the flow. Also we can see that $P_1, P_2 \in M_{inv}$. The motion restricted to M_{inv} represents the planar motion case of the rigid body.

Now we are able to define the unperturbed and perturbed systems.

The unperturbed system is given by (11) when the CM has a circular motion. The perturbed system is (11) when the CM has an elliptic motion. Thus, the eccentricity ε plays the rôle of a perturbation parameter. Using (3), the system (11) turns into:

$$x' = f(x) + \varepsilon g(x, v, \varepsilon) \tag{14}$$

where $x^T = (\varphi, \theta, p_\Phi, p_\theta) \in T^*S^2$ is the state vector and f, g are vector fields given by:

$$f(x) = \begin{bmatrix} \frac{2p^2}{CI_1}\frac{p_\Phi}{sin^2\theta} - 2 \\ \frac{p^2}{CI_1}p_\theta \\ \frac{3GM}{2pC}(I_1 - I_3)sin^2\theta \sin\varphi \\ \frac{p^2}{CI_1}\frac{p_\Phi{}^2 \cos\theta}{sin^3\theta} + \frac{3GM}{2pC}(I_1 - I_3)\sin 2\theta \frac{1-\cos\varphi}{2} \end{bmatrix} \tag{15}$$

and:

$$g(x, v, \varepsilon) = \begin{bmatrix} -\frac{2p^2}{CI_1}\frac{p_\Phi \cos v(2+\varepsilon\cos v)}{sin^2\theta(1+\varepsilon\cos v)^2} \\ -\frac{p^2}{CI_1}\frac{p_\theta \cos v(2+\varepsilon\cos v)}{(1+\varepsilon\cos v)^2} \\ \frac{3GM}{2pC}(I_1 - I_3)sin^2\theta\cos v \sin\varphi \\ -\frac{p^2}{CI_1}\frac{p_\Phi{}^2\cos\theta\cos v(2+\varepsilon\cos v)}{sin^3\theta(1+\varepsilon\cos v)^2} + \frac{3GM}{2pC}(I_1 - I_3)\sin 2\theta \cos v\frac{1-\cos\varphi}{2} \end{bmatrix} \tag{16}$$

Now we analyze the unperturbed system. Consider that the CM is moving on a circular orbit given by $r = p$. Firstly we change the variables as follows:

$$x_1 = 2(\Phi - v) \qquad x_2 = \theta$$

$$p_1 = \frac{r^2}{2CI_1} p_\Phi \qquad p_2 = \frac{r^2}{CI_1} p_\theta$$

and then (11) is brought into the following form (note that $GMp = C^2$ conform (5)):

$$\begin{aligned} \frac{dx_1}{dv} &= \frac{4p_1}{\sin^2 x_2} - 2 \\ \frac{dx_2}{dv} &= p_2 \\ \frac{dp_1}{dv} &= \frac{3}{4}\frac{I_1 - I_3}{I_1} \sin^2 x_2 \sin x_1 \\ \frac{dp_2}{dv} &= \frac{4p_1^2 \cos x_2}{\sin^3 x_2} + \frac{3}{2}\frac{I_1 - I_3}{I_1} \sin 2x_2 \frac{1 - \cos x_1}{2} \end{aligned} \tag{17}$$

which is Hamiltonian with:

$$\begin{aligned} &\bar{H}(x_1, x_2; p_1, p_2) = \frac{2p_1^2}{\sin^2 x_2} + \frac{p_2^2}{2} - \frac{3}{2}\frac{I_1 - I_3}{I_1} \sin^2 x_2 \frac{1 - \cos x_1}{2} - 2p_1 \\ &\bar{H} : T^* S^2 \to \mathbf{R} \end{aligned} \tag{18}$$

The zeros P_1, P_2 given by (12) become relative equilibria for the system. These are, in the new variables:

$$Q_1 = (0, \frac{\pi}{2}, \frac{1}{2}, 0) \qquad Q_2 = (\pi, \frac{\pi}{2}, \frac{1}{2}, 0)$$

The differential of the vector field (11) (i.e. of the right-hand side) has the form:

$$Df|_{(x_1, x_2, p_1, p_2)} = \begin{bmatrix} 0 & -\frac{8p_1 \cos x_2}{\sin^3 x_2} & \frac{4}{\sin^2 x_2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3}{4}\frac{I_1 - I_3}{I_1} \sin^2 x_2 \cos x_1 & \frac{3}{4}\frac{I_1 - I_3}{I_1} \sin 2x_2 \sin x_1 & 0 & 0 \\ \frac{3}{4}\frac{I_1 - I_3}{I_1} \sin 2x_2 \sin x_1 & * & \frac{8p_1 \cos x_2}{\sin^3 x_2} & 0 \end{bmatrix} \tag{19}$$

where:

$$* = -\frac{4p_1^2}{\sin^2 x_2} - \frac{12p_1^2 \cos^2 x_2}{\sin^4 x_2} + 3\frac{I_1 - I_3}{I_1} \cos 2x_2 \frac{1 - \cos x_1}{2}$$

At Q_1 it becomes:

$$Df(Q_1) = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3}{4}\frac{I_1 - I_3}{I_1} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{20}$$

while at Q_2 we get:

$$Df(Q_2) = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{3}{4}\frac{I_1 - I_3}{I_1} & 0 & 0 & 0 \\ 0 & -1 - 3\frac{I_1 - I_3}{I_1} & 0 & 0 \end{bmatrix} \tag{21}$$

The characteristic polynomials are:

$$\begin{array}{l} p_{Q_1}(s) = (s^2+1)(s^2 - 3\frac{I_1-I_3}{I_1}) \\ p_{Q_2}(s) = (s^2 + 3\frac{I_1-I_3}{I_1})(s^2+1+3\frac{I_1-I_3}{I_1}) \end{array} \tag{22}$$

Now we see that depending on the value of $a = 3\frac{I_1-I_3}{I_1}$ we have different types of equilibria:

· for $a > 0$, Q_1 is center-saddle and Q_2 is center;
· for $-1 < a < 0$, Q_1 is center and Q_2 is center-saddle;
· for $a < -1$, Q_1 is center and Q_2 is saddle;

We turn now to the restricted system on the invariant manifold given in Lemma 1. We shall prove that for any value of $a \neq 0$ we have two homclinic orbits to a saddle point and these connections will be preserved under the perturbation. Then we shall return to the full system on T^*S^2 (17) and we shall prove that the stable and unstable manifolds to the saddle point (Q_2, when $a < -1$) are candidates for a hyperbolic structure and consequently for a "global" chaotic motion and this will give us the analytic nonintegrability of the Hamiltonian system.

Now let us consider the restriction of (11) to the invariant manifold M_{inv} given in Lemma 1. We set $\theta = \frac{\pi}{2}$ and $p_\theta = 0$ and we obtain:

$$\begin{array}{rcl} \frac{dp_\Phi}{dv} & = & \frac{3GM}{2rC}(I_1 - I_3)\sin\varphi \\ \frac{d\varphi}{dv} & = & \frac{2r^2}{CI_1}p_\Phi - 2 \end{array} \tag{23}$$

equivalent to the following second order differential equation:

$$\frac{d^2\varphi}{dv^2} - 3\frac{GMr}{C^2}\frac{I_1 - I_3}{I_1}\sin\varphi = \frac{2}{r}\frac{dr}{dv}(2 + \frac{d\varphi}{dv}) \tag{24}$$

This system is still a Hamiltonian system obtained from (9) by setting $\theta = \frac{\pi}{2}$ and $p_\theta = 0$. We rewrite (24) as a system of 2 first differential equations using as state variables: if $I_1 < I_3$ $y_1 = \varphi$ else $y_1 = \varphi + \pi$ and $y_2 = \frac{d\varphi}{dv}$. We get:

$$\begin{array}{rcl} \frac{dy_1}{dv} & = & y_2 \\ \frac{dy_2}{dv} & = & -3\frac{GMr}{C^2}|\frac{I_1-I_3}{I_1}|\sin y_1 + \frac{2}{r}\frac{dr}{dv}(2 + y_2) \end{array} \tag{25}$$

We use the decomposition given in (14) using (3) and (5). We get:

$$\begin{array}{rcl} \frac{dy_1}{dv} & = & y_2 \\ \frac{dy_2}{dv} & = & -3|\frac{I_1-I_3}{I_1}|\sin y_1 + \varepsilon(3|\frac{I_1-I_3}{I_1}|\sin y_1 \frac{\cos v}{1+\varepsilon\cos v} + 2(2+y_2)\frac{\sin v}{1+\varepsilon\cos v}) \end{array} \tag{26}$$

We can denote $\Omega^2 = 3|\frac{I_1-I_3}{I_1}| > 0$ and we see that the unperturbed system corresponds to $\frac{d^2y_1}{dv^2} + \Omega^2\sin y_1 = 0$ which is a pendulum-like equation. We know that this equation yields two homoclinic connections at $y_1 = \pi,\ y_2 = 0$. The homoclinic connections are given by:

$$\begin{aligned} y_1^0(v) &= \pm 2\, arctan(sinh(\Omega v)) \\ y_2^0(v) &= \pm 2\Omega\, sech(\Omega v) \end{aligned} \tag{27}$$

where $+$ stands for the upper homoclinic connections (in y_1, y_2 plane) and $-$ corresponds to the lower branch of the homoclinic orbits.

Now we are interested to see if these homoclinic connections are preserved under the perturbation. The answer is given by the following result:

LEMMA 2. *For any* $\Omega > 0$ *the system* (26) *has infinitely many transversal homoclinic orbits for any* $\varepsilon \in (0, 1)$ *excepting, at most, for a finite number of values.* □

This Lemma can be proved by using the Melnikov's function and Smale-Birkhoff Theorem; we refer the reader to (Teofilatto *et al.*, 1992) and (Burov, 1987) (conform to (Teofilatto *et al.*, 1992)).

Now we can return to (17) which describes the unperturbed system on T^*S^2. We consider only the case $a < -1$, or equivalently $I_3/I_1 > 4/3$. The unperturbed system has at Q_2 a saddle point and then two invariant 2-dimensional manifolds pass through Q_2, the stable and unstable manifolds. From the above discussion we know there exist two homoclinic connections. Then the intersection of the stable and unstable manifolds is non empty and even more, it contains at least two 1-dimensional curves. In order to obtain transversal intersection of these manifolds (for the perturbed system), we need to prove that the intersection is precisely of dimension 1 and this is offered by the following Lemma:

LEMMA 3. *For the unperturbed system* (17) *(i.e.* $\varepsilon = 0$*) with* $I_3/I_1 > 4/3$ *consider a point* q^0 *on the homoclinic connections* (27) *away from* Q_2*. Let us denote by* $W^{s,u}$ *the stable/unstable manifolds passing through* Q_2*. Then* $q^0 \in W^s \cap W^u$ *and* $\dim(T_{q^0}W^s + T_{q^0}W^u) = 3$ *for all values of* Ω *excepting for at most 1 value.* □

The proof is based on two steps: firstly we find the first correction to the stable and unstable manifold around the relative equilibrium Q_2; then we solve asymptotically the first variational system that transports the tangent vectors along the homoclinic orbits. The critical value of Ω is found to be $\Omega_c = 1.70557$ and this happens only for the upper branch of the homoclinic orbits. The details are presented in section 5.

4. Statement of the Main Results

In this section we present the conclusion of the Lemmas 2 and 3 from the previous section.

As we have said, Lemma 2 has been proved in some other papers (see (Teofilatto *et al.*, 1992) and (Burov, 1987)). We give here just a briefly interpretation of the symbolic dynamics associated to the chaotic motion that occurs because of the existence of transversal homoclinic points. This idea is taken from a lecture given by Professor P.Holmes at Princeton University.

THEOREM 4. *For any* $\Omega > 0$ *and almost any* $\varepsilon \in (0,1)$ *(except, at most, for a finite number of values) the planar attitude motion of the rigid body (i.e. the motion restricted to* M_{inv}*) has a chaotic behaviour in the following sense: for any sequence of integers* $s = (s_k)_{k\in\mathbf{Z}}$, $s_z \in \mathbf{Z}$ *there exists a sequence of increasing numbers* $(t_k)_{k\in\mathbf{Z}}$, $t_k < t_{k+1}$, $t_k \in \mathbf{R}$ *and a trajectory of* (23) *such that :* $\varphi(t_k) = 2\pi s_k$, *for any* k. □

This form of the chaotic motion means that the rigid body can rotate for an arbitrary number of times in a sense, then stop and rotate in the opposite sense for another arbitrary number of times and so on. The only problem is to find the initial condition. The initial condition is found by using the horseshoe associated to the transversal homoclinic points. This standard construction is presented in many papers; we refer the reader, for instance, to (Moser, 1973),(Guckenheimer *et al.*, 1993) or (Xia, 1992).

Lemma 3 proves that, in the unperturbed case, the intersection between stable and unstable manifolds contains only the homoclinic orbit (27) and this is included in the invariant manifold M_{inv} given in Lemma 1. Then, for almost any $\varepsilon > 0$ the intesection between stable and unstable manifolds of the perturbed system will contain at most some 1-dimensional curves included in M_{inv}. But, as we have proved in Lemma 2, the homoclinic orbit will break transversally under the perturbation into an infinite set of homoclinic points. Thus the stable and unstable manifolds in the 4-dimensional phase space will contain only these points and then they will intersect transversally.

This transversality gives us the analytic nonintegrability of the system (we refer the reader to (Kozlov, 1983) for an extensive survey on nonintegrability of Hamiltonian systems). Here we shall state a result about non-existence of 2 analytic, independent first integrals.

Suppose we have a periodic and analytic Hamiltonian $H_\varepsilon : \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}$ dependent analytic on a small parameter $\varepsilon > 0$ ($H_\varepsilon(x,p,t+T) = H_\varepsilon(x,p,t)$). Consider the Hamiltonian system:

$$\begin{aligned} \dot{x} &= \frac{\partial H_\varepsilon}{\partial p} \\ \dot{p} &= -\frac{\partial H_\varepsilon}{\partial x} \end{aligned} \tag{28}$$

and associate to it the Poincaré return map:

$$P_{t_0}^{\varepsilon} : (x_1,p_1) \mapsto P_{t_0}^{\varepsilon}(x_1,p_1) = (x_2,p_2)$$

where x_2, p_2 is the solution of (28) at $t_0 + T$ when at t_0 $(x,p) = (x_1,p_1)$.

A function $F^\varepsilon : \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}$ periodic in time ($F^\varepsilon(x,p,t+T) = F^\varepsilon(x,p,t)$) and depending on ε as a formal power series:

$$F^\varepsilon(x,p,t) = \sum_{i\geq 0} \varepsilon^i F^i(x,p,t)$$

is said to be an *analytic first integral* if:

1) $F^i : \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}$ are analytic ;

2) $F^\varepsilon(P^\varepsilon_{t_0}(x,p),t_0) = F^\varepsilon(x,p,t_0)$, for any $(x,p) \in \mathbf{R}^{2n}$ and $t_0 \in [0,T]$

A set of n analytic first integrals $F^\varepsilon_1, \ldots, F^\varepsilon_n : \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}$ is said to be *independent* if the level set:

$$M_c(t_0) = \{(x,p) \in \mathbf{R}^n \mid F^\varepsilon_k(x,p,t_0) = c_k \ , \ 1 \leq k \leq n \ \}$$

does not include any manifold of dimension higher than n (the set $M_c(t_0)$ is an analytic set and, because of Lojaciewicz's result, that we shall state and use in a moment, it can be written as local finite union of analytic manifolds). Now we can state our nonintegrability result:

THEOREM 5. *Consider a spinless, axisymmetric rigid body lying in a central gravitational field, whose attitude motion dynamics is given by* (11). *If* $I_3/I_1 > 4/3$ *then there do not exist 2 analytic, independent first integrals, and then the system is analytical nonintegrable.* □

We shall give a straightforward proof of this result (as well as for any system where we have a transversal intersection of the stable and unstable manifolds) based on the λ-Lemma and Lojaciewicz's Structure Theorem for Real Analytic Manifolds. Another proof can be done using the (Kozlov, 1983) paper, by noting that the union of stable and unstable manifolds $W^s_\varepsilon \cup W^u_\varepsilon$ is a key set, in the terminology of the aforementioned paper. We recall now the two results; from the Lojaciewicz's Structure Theorem we present only the result that we are using.

λ-Lemma (see (Palis, 1969)) Let f be a $\mathcal{C}^1$ diffeomorphism of $\mathbf{R}^n$ with a hyperbolic fixed point p having s and u dimensional stable and unstable manifolds ($s+u=n$), and let D be a u-dimensional disk in $W^u(p)$. Let Δ be a u-dimensional disk meeting $W^s(p)$ transversely at some point q. Then $\bigcup_{n\geq 0} f^n(\Delta)$ contains u-dimensional disks arbitrarily close to D. □

Lojaciewicz's Structure Theorem for Real Analytic Manifolds (see (Krantz *et al.*, 1992) for the complete statement, pp.154) Let $\Phi(x_1, \ldots, x_n)$ be a real nontrivial analytic function in a neighborhood of the origin. Then there exist numbers $\delta_j > 0, j = 1, ..., n$ so that the set:

$$Z = \{x \in \mathbf{R}^n | \ |x_j| < \delta_j, \forall j \ and \ \Phi(x) = 0 \ \}$$

has a decomposition:

$$Z = V^{n-1} \cup \cdots \cup V^0$$

The set V^0 is either empty or consists of the origin alone. For $1 \leq k \leq n-1$ we may write V^k as a finite, disjoint union of k-dimensional submanifolds (in the full statement, an explicit description of these manifolds is given). □

Now we prove Theorem 5. Suppose there are 2 analytic, independent first integrals, say F_1^ε and F_2^ε. Suppose we have fixed t_0 and denote by $P_{t_0}^\varepsilon$ the Poincaré map. Then, on stable manifold they must be constant. The same thing happens on the unstable manifold. Because the stable and unstable manifolds intersect the values of F_1^ε, respectively F_2^ε, must be the same on these manifolds, that is:

$$F_1^\varepsilon(W^u) = F_1^\varepsilon(W^s) = c_1 \quad F_2^\varepsilon(W^u) = F_2^\varepsilon(W^s) = c_2$$

Now, pick a point $s_0 \in W^u$ and consider q a transversal intersection point between W^s and W^u, different from the fixed point of $P_{t_0}^\varepsilon$ (such a point exists because of Lemmas 2 and 3). Let Δ be a 2-dimensional disk in W^u containing q as in λ-Lemma. Then, for any neighborhood of s_0 there exists an integer $n > 0$ such that $(P_{t_0}^\varepsilon)^n(\Delta)$ intersects nonempty the neighborhood. Now we apply the Lojaciewicz's Theorem to:

$$\Phi(x) = (F_1^\varepsilon(x+s_0) - c_1)^2 + (F_2^\varepsilon(x+s_0) - c_2)^2$$

Denote by

$$Z_\delta = \{x \in T^*S^2 | \parallel x \parallel \leq \delta \ , \ \Phi(x) = 0\}$$

which is the intersection between the level set $Z = \{x \in T^*S^2 | \Phi(x) = 0\}$ and the ball $B_\delta(s_0) = \{x \in T^*S^2 | \parallel x - s_0 \parallel \leq \delta\}$. Then $W^u \cap B_\delta(s_0)$ and $W^s \cap B_\delta(s_0)$ are both included in Z_δ. Particularly we are interested in the inclusion $W^u \cap B_\delta(s_0) \subset Z_\delta$. Now we have a decomposition of Z_δ into a union of manifolds of dimension 0 (the point s_0), 1 and 2 (dimensions higher than 2 are forbidden by the condition that F_1^ε and F_2^ε are independent). Now, if we look to the union of manifolds of dimension 2 we see that here must lie an infinite sequence of submanifolds of the form $(P_{t_0}^\varepsilon)^n(\Delta) \cap B_\delta(s_0)$, for some n. Then we conclude that Z_δ is not a local finite union of manifolds and this proves the contradiction. So, our assumption of the existence of 2 analytic, independent first integrals is false.

5. Proof of Lemma 3.3

The situation is now the following: we have the unperturbed system given by (17) and we are in the case when $a = 3\frac{I_1 - I_3}{I_1} < -1$. This means that Q_2 is a saddle point and then it is a hyperbolic equilibrium in T^*S^2 for (17). We know from the Stable Manifold Theorem (see (Kelley, 1967)) that two invariant 2-dimensional manifolds pass through Q_2 tangent to, respectivelly, the stable space and unstable space of the linearized system (21). Each of them contains the homoclinic connections (27) so that their intersection, except for Q_2, is not empty. We want to prove that, along of these homoclinic orbits there are three independent vectors tangent to the union of the manifolds (i.e. two of them to a manifold and the third vector to the other manifold). We choose one of the three independent vectors to be the tangent vector to the homoclinic orbits at that point (q^0). This is tangent to both invariant manifolds. We shall prove that taking two other vectors to the unstable, respectively, stable manifolds near Q_2, they are transported by the flow through the homoclinic orbit forward, respectively, backward at the same point into two independent vectors. For, we need two facts: firstly we have to know the first correction of the tangent spaces to the stable/unstable manifolds near Q_2 and secondly, we have to find an asymptotic approximation for the transports along the homoclinic orbits of a tangent vector (i.e. an asymptotic expansion of the solution of the first variational equation).

To simplify the calculus we translate the equilibrium point Q_2 into the origin by changing the variables as follows:

$$\xi_1 = x_1 - \pi \ , \ \xi_2 = x_2 - \frac{\pi}{2} \ , \ \xi_3 = p_1 - \frac{1}{2} \ , \ \xi_4 = p_2$$

Then, the system (17) becomes:

$$\begin{aligned} \xi_1{}' &= \tfrac{4\xi_3+2}{\cos^2\xi_2} - 2 \\ \xi_2{}' &= \xi_4 \\ \xi_3{}' &= \tfrac{1}{4}\Omega^2\cos^2\xi_2 \sin\xi_1 \\ \xi_4{}' &= -\tfrac{4(\xi_3+\frac{1}{2})^2 \sin\xi_2}{\cos^3\xi_2} + \tfrac{1}{2}\Omega^2 \sin(2\xi_2)\tfrac{1+\cos\xi_1}{2} \end{aligned} \tag{29}$$

whose Hamiltonian is:

$$\begin{aligned} &H(\xi_1,\xi_2,\xi_3,\xi_4) = \bar{H}(\xi_1+\pi,\xi_2+\tfrac{\pi}{2},\xi_3+\tfrac{1}{2},\xi_4) = \\ &\tfrac{2{\xi_3}^2+2\xi_3+\frac{1}{2}}{\cos^2\xi_2} + \tfrac{{\xi_4}^2}{2} - 2\xi_3 - 1 + \tfrac{1}{2}\Omega^2\cos^2\xi_2\tfrac{1+\cos\xi_1}{2} \end{aligned} \tag{30}$$

and the equilibrium point is now the origin $(\xi_1,\xi_2,\xi_3,\xi_4) = (0,0,0,0)$.

We compute now the first correction to the tangent spaces to the invariant manifolds. For, we use a very nice result about these manifolds, proved in (Schaft, 1991) or see also (Kozlov, 1983). The result says that both the stable and unstable manifolds are Lagrange submanifolds (see (Abraham *et al.*, 1978) for details on Lagrange submanifolds). Then, there exist two analytic scalar functions $V^{s,u}$: $D \subset \mathbf{R}^2 \to \mathbf{R}$, $(\xi_1,\xi_2) \mapsto V^{s,u}(\xi_1,\xi_2)$ defined on a neighborhood of the origin that satisfy the Hamilton-Jacobi equation:

$$H(\xi_1,\xi_2,\nabla V^{s,u}) = H(0) \tag{31}$$

and the graphs of the gradient of these functions are exactly the local stable and , respectively, unstable manifolds of (29), providing that these two manifolds can be parametrized by using the first two coordinates ξ_1 and ξ_2 (this is the disconjugacy condition of the Hamiltonian system with respect to the stable and unstable solutions of (31)). We shall use the equation (31) to find the first correction to the quadratic terms of $V^{s,u}$ (i.e. the third order terms).

Firstly we check the disconjugacy. For we recall $Df(Q_2)$ given in (21). We know from (22) that the spectrum of the linearized system is given by $Spec = \{\Omega, -\Omega, \sqrt{\Omega^2-1}, -\sqrt{\Omega^2-1}\}$. The corresponding eigenvectors are:

· the unstable space:

$$\lambda_1 = \Omega \quad , \quad v_1^T = (1, 0, \frac{\Omega}{4}, 0)$$
$$\lambda_2 = \sqrt{\Omega^2-1} \quad , \quad v_2^T = (0, 1, 0, \sqrt{\Omega^2-1})$$

· the stable space:

$$\lambda_3 = -\Omega \quad , \quad v_3^T = (1, 0, -\frac{\Omega}{4}, 0)$$
$$\lambda_4 = -\sqrt{\Omega^2-1} \quad , \quad v_4^T = (0, 1, 0, -\sqrt{\Omega^2-1})$$

Now it is obvious that the projections of both E^u and E^s, the unstable and stable spaces into the 2-dimensional space spanned by $e_1^T = (1,0,0,0)$ and $e_2^T = (0,1,0,0)$, are of dimensions 2. Either from the geometric theory of the Algebraic Riccati Equations (see (Shayman, 1983) for details) or by straightforward calculus we check that the quadratic terms in $V^{s,u}$ are given by:

$$V^{s,u}_{\leq 2} = \xi^T X^{s,u} \xi$$

where $\xi^T = (\xi_1, \xi_2)$ and:

$$X^s = \begin{bmatrix} -\frac{\Omega}{4} & 0 \\ 0 & -\sqrt{\Omega^2-1} \end{bmatrix} \quad , \quad X^u = \begin{bmatrix} \frac{\Omega}{4} & 0 \\ 0 & \sqrt{\Omega^2-1} \end{bmatrix} \tag{32}$$

Now, if we keep up to the third term in $V^{s,u}$ we obtain:

$$V^s_{\leq 3}(\xi_1, \xi_2) = \tfrac{1}{2}\xi^T X^s \xi + third\_order\_terms =$$
$$= -\tfrac{\Omega}{8}{\xi_1}^2 - \tfrac{\sqrt{\Omega^2-1}}{2}{\xi_2}^2 + b_1{\xi_1}^3 + b_2{\xi_1}^2\xi_2 + b_3\xi_1{\xi_2}^2 + b_4{\xi_2}^3$$

and analogously for $V^u_{\leq 3}(\xi_1, \xi_2)$. We have now to introduce in (31) and identify b_1, b_2, b_3, b_4 by expanding up to the third order. The expansion of H up to the third order has the form:

$$H_{\leq 3}(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\Omega^2-1}{2} - \frac{1}{8}\Omega^2{\xi_1}^2 - \frac{\Omega^2-1}{2}{\xi_2}^2 + 2{\xi_3}^2 + \frac{1}{2}{\xi_4}^2 + 2{\xi_2}^2\xi_3$$

and the solutions for $V^s_{\leq 3}$ and $V^u_{\leq 3}$ are:

$$V^u(\xi_1, \xi_2) = \frac{\Omega}{8}{\xi_1}^2 + \frac{1}{2}\sqrt{\Omega^2-1}{\xi_2}^2 - \frac{1}{2}\frac{\Omega}{\Omega + 2\sqrt{\Omega^2-1}}\xi_1{\xi_2}^2$$

$$V^s(\xi_1,\xi_2) = -\frac{\Omega}{8}{\xi_1}^2 - \frac{1}{2}\sqrt{\Omega^2-1}{\xi_2}^2 - \frac{1}{2}\frac{\Omega}{\Omega+2\sqrt{\Omega^2-1}}\xi_1{\xi_2}^2$$

Now, the invariant manifolds are given by:

$$(\xi_1,\xi_2) \longrightarrow (\xi_1,\xi_2,\frac{\partial V^{u,s}}{\partial \xi_1},\frac{\partial V^{u,s}}{\partial \xi_2})$$

and the approximating tangent vectors to these manifolds, computed on the homoclinic orbits (where $\xi_2 = 0$) are given by:

· for the unstable manifold:

$$\begin{array}{c} X_1 = \frac{\partial}{\partial \xi_1} + \frac{\Omega}{4}\frac{\partial}{\partial \xi_3} \\ X_2 = \frac{\partial}{\partial \xi_2} + (\sqrt{\Omega^2-1} - \frac{\Omega}{\Omega+2\sqrt{\Omega^2-1}}\xi_1)\frac{\partial}{\partial \xi_4} \end{array} \tag{33}$$

· for the stable manifold:

$$\begin{array}{c} X_3 = \frac{\partial}{\partial \xi_1} - \frac{\Omega}{4}\frac{\partial}{\partial \xi_3} \\ X_4 = \frac{\partial}{\partial \xi_2} - (\sqrt{\Omega^2-1} - \frac{\Omega}{\Omega+2\sqrt{\Omega^2-1}}\xi_1)\frac{\partial}{\partial \xi_4} \end{array} \tag{34}$$

and these hold only for $|\xi_1| + |\xi_2|$ small enough. Now it is straightforward to see that X_1 and X_3 are tangent to the homoclinic orbits at the origin. Thus, what we have to do is to prove that X_2 is not transported along the homoclinic orbits into X_4.

It is known that a tangent vector is transported along a curve via the first variational system which is a linear time-varying system of the form:

$$z' = Df|_{\varphi(v)}z \tag{35}$$

For our system (17), the differential of the vector field along the homoclinic orbits has the form (recall $x_2 = \frac{\pi}{2}$ and $p_2 = 0$):

$$Df|_{(x_1(v),\frac{\pi}{2},p_1(v),0)} = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{4}\Omega^2\cos x_1 & 0 & 0 & 0 \\ 0 & -4p_1^2 + \Omega^2\frac{1-\cos x_1}{2} & 0 & 0 \end{bmatrix}$$

We see that, if $z^T = (z_1, z_2, z_3, z_4)$ then (35) decomposes into two 2-dimensional systems:

$$\begin{array}{rcl} z_1' &=& 4z_3 \\ z_3' &=& -\frac{1}{4}\Omega^2\cos x_1\, z_1 \end{array} \tag{36}$$

and:

$$\begin{array}{rcl} z_2' &=& z_4 \\ z_4' &=& (-4p_1^2 + \Omega^2\frac{1-\cos x_1}{2})z_2 \end{array} \tag{37}$$

The initial condition for the forward transport is given by X_2: $z_1 = 0$, $z_2 = 1$, $z_3 = 0$, $z_4 = \sqrt{\Omega^2-1} - \frac{\Omega}{\Omega+2\sqrt{\Omega^2-1}}\xi_1$ and then only (37) is the system to be analyzed (actually (36) gives the transport of the tangent vector to the homoclinic orbits along the homoclinic orbits and the solution is obvious $z_1 = x_1'(v)$, $z_3 = p_1'(v)$). We rewrite (37) as a second-order differential equation:

$$z_2'' + (4p_1^2 - \Omega^2 \frac{1-\cos x_1}{2})z_2 = 0$$

We use the explicit form (27) of the homoclinic orbits (recall now $I_1 < I_3$ and then $x_1 = y_1^0$, $p_1 = \frac{1}{2} + \frac{1}{4}y_2^0$) and we get:

$$z_2'' + h(v)z_2 = 0 \qquad \text{where } h(v) = 1 \pm 2\Omega sech(\Omega v) + \Omega^2(1 - 2tanh^2(\Omega v)) \tag{38}$$

Now we change the variable $v \to u = tanh(\frac{\Omega v}{2})$. Then (38) becomes:

$$\frac{\Omega^2}{4}(1-u^2)^2 \frac{d^2 z_2}{du^2} - \frac{\Omega^2}{2}u(1-u^2)\frac{dz_2}{du} + k(u)z_2 = 0 \tag{39}$$

with:

$$k(u) = 1 \pm 2\Omega \frac{1-u^2}{1+u^2} + \Omega^2 \frac{1-6u^2+u^4}{(1+u^2)^2} \tag{40}$$

and the interval of analysis is $(-1, 1)$. The initial condition, which is given by the tangent vector to the unstable manifold, corresponds to $u \to -1$. Let's consider $u = -1 + \varepsilon$ and try to evaluate z_4 up to order ε. Firstly, we have to find ξ_1. We have:

$$\xi_1 = x_1 - \pi = (\pm 2arctan(sinh(\Omega v)) - \pi) mod\ 2\pi$$

If we consider $tanh\frac{\Omega v}{2} = -1 + \varepsilon$ and expand ξ_1 we get:

$$\xi_1 = \pm 2\varepsilon + \mathcal{O}(\varepsilon^2)$$

Now, we need $\frac{dz_2}{du}$. We know that $\frac{dz_2}{dv} = z_4$, then:

$$\frac{dz_2}{du} = \frac{dv}{du}\frac{dz_2}{dv} = \frac{2}{\Omega}\frac{1}{1-u^2}z_4$$

and using the initial condition for z_4, at $u = -1 + \varepsilon$ we get:

$$\frac{\frac{dz_2}{du}}{z_2}|_{u=-1+\varepsilon} = \frac{\sqrt{\Omega^2-1}}{\Omega}\frac{1}{\varepsilon} \mp \frac{2}{\Omega + 2\sqrt{\Omega^2-1}} + \mathcal{O}(\varepsilon) \tag{41}$$

Now we analyze the asymptotic solution of (39) near to $u_0 = -1$. Firstly we see that both $u_0 = -1$ and $u_1 = 1$ are regular singular points (see (Bender, 1978) for a general tratement of asymptotic approximations). We look for an asymptotic of the form $z_2 \sim (1+u)^\alpha$ (near u_0). Plugging into (39) and setting $u = -1$ we get for α the equation:

$$\alpha^2\Omega^2 = -k(-1) \;\Rightarrow\; \alpha = \pm\frac{1}{\Omega}\sqrt{-k(-1)} = \pm\frac{\sqrt{\Omega^2-1}}{\Omega}$$

The Frobenius solution of the equation has then the form: $z_2 = (1+u)^\alpha P(u)$ where $P(u)$ is a polynomial in u. We keep only the first two terms from $P(u)$ and we get:

$$z_2 \sim (1+u)^\alpha (C_0 + C_1 u) \tag{42}$$

We compute C_0 and C_1 by requiring the initial condition (41). We obtain:

$$\frac{\frac{dz_2}{du}}{z_2} = \frac{\alpha}{1+u} + \frac{C_1}{C_0 + C_1 u} \overset{u=-1+\varepsilon}{\sim} \frac{\alpha}{\varepsilon} + \frac{C_1}{C_0 - C_1} + \mathcal{O}(\varepsilon) \tag{43}$$

By comparing (43) with (41) we get that $\alpha = \frac{\sqrt{\Omega^2-1}}{\Omega}$ and:

$$K = \frac{C_1}{C_0} = \mp\frac{2}{\Omega \mp 2 + 2\sqrt{\Omega^2-1}} \tag{44}$$

Then, at $u = 0$ we get:

$$\frac{\frac{dz_2}{du}}{z_2}|_{u=0} = \alpha + K \tag{45}$$

Similarly we can compute the transport of X_2 backward in time, from $u = 1 - \varepsilon$ to $u = 0$. We get:

$$\frac{\frac{dz_2}{du}}{z_2} = -\alpha - K \tag{46}$$

with the same expressions for α and K as above.

Thus, the condition that at $u = 0$ to have three independent vectors is that (45) and (46) do not coincide, that is:

$$\alpha + K \neq -\alpha - K \tag{47}$$

If the above condition is fulfilled, then the tangent vector X_2 is everywhere independent of X_4 and this proves the Lemma.

For the lower branch, the condition (47) takes the form:

$$\frac{\sqrt{\Omega^2-1}}{\Omega} + \frac{2}{\Omega + 2 + 2\sqrt{\Omega^2-1}} \neq 0 \tag{48}$$

which is always true for $\Omega > 1$. For the upper branch the condition (47) turns into:

$$\frac{\sqrt{\Omega^2-1}}{\Omega} - \frac{2}{\Omega - 2 + 2\sqrt{\Omega^2-1}} \neq 0 \tag{49}$$

which has a root at $\Omega \simeq 1.70557$. To completely solve the problem for the upper branch, one must go to higher order approximations for $V^{s,u}$ in (31) and $z_2(u)$ in (42), but, for genericity, this result is enough.

6. Conclusions

In this paper we have obtained a analytic nonintegrability result of a Hamiltonian system. The problem was to study the rotation motion of a rigid body in a central gravitational field. To obtain the result, four assumptions were made.

The first assumption concerns the motion of the center of mass of the rigid body and it is supposed to be undisturbed by the rotation motion. For instance this is true if the ratio between the dimension of the rigid body and the distance up to the center of attraction is much less than 1.

The second assumption is less critical but is made in order to avoid messy calculus. Under this assumption we neglect the higher-order terms in the interaction Hamiltonian.

The third assumption, namely the axisymmetry of the rigid body, is made in order to progress in description. The condition $I_3/I_1 > 4/3$ is essential for the hyperbolicity of P_2 and for the transversal intersection of the stable/unstable manifolds of the perturbed system.

The fourth assumption, i.e. to consider a spinless top, is a technical one. Assuming a spinless top we are able to find homoclinic orbits and then to construct the horseshoes.

Under these assumptions we have proved that our problem gives rise to a time-varying Hamiltonian system on a two-dimensional sphere. The eccentricity of the orbit of the CM plays the rôle of a perturbation parameter. The unperturbed system (i.e. corresponding to a circular orbit of the CM) has a hyperbolic saddle point whose stable and unstable manifolds intersect only along the homoclinic connexions. The perturbation preserves the homoclinic connections, yielding to transversal homoclinic orbits. Then the stable and unstable manifolds intersect transversaly in T^*S^2. This is immediately connected with some chaotic behaviour of the flow and, especially, with analytic nonintegrability of the system.

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