Equivalence of Reconstruction from the Absolute Value of the Frame Coefficients to a Sparse Representation Problem

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***Abstract*— The purpose of this note is to prove, for real frames, that signal reconstruction from the absolute value of the frame coefficients is equivalent to solution of a sparse signal optimization problem, namely a minimum ℓ^p (quasi)norm over a linear constraint. This linear constraint reflects the coefficients relationship within the range of the analysis operator.**

***Index Terms*— frames, nonlinear processing, sparse representation**

I. Introduction

In our previous paper [2] we considered the problem of signal reconstruction from the absolute value of its coefficients in a redundant representation. We obtained necessary and sufficient conditions for perfect reconstruction up to a constant phase factor. In the finite dimensional setting, a **frame** for a Hilbert space is just a set of vectors spanning the Hilbert space. For real valued signals and real valued transformations we obtained the following result

Theorem 1.1: Let $\mathbb{M}:\mathbb{R}^N/\{+1,-1\}\ \rightarrow\mathbb{R}^M$ be defined by $\mathbb{M}(x)=\{|\langle x,f_k\rangle|\}_{1\leq k\leq M}$, where $\mathcal{F}=\{f_1,f_2,\ldots,f_M\}$ spans $\mathbb{R}^N$. Then:

1) If $M\geq 2N-1$ then for a generic frame $\mathcal{F}$, the map $\mathbb{M}$ is injective;
2) If $\mathbb{M}$ is injective, then $M\geq 2N-1$;
3) If $M=2N-1$ then $\mathbb{M}$ is injective if and only if every N-element subset of $\mathcal{F}$ is linearly independent.
4) $\mathbb{M}$ is injective if and only if for every subset $\mathcal{G}\subset\mathcal{F}$, either $\mathcal{G}$ or $\mathcal{F}\setminus\mathcal{G}$ spans $\mathbb{R}^N$.
5) If $M>N$ then for a generic frame $\mathcal{F}$, the set of points $x\in\mathbb{R}^N/\{+1,-1\}$ so that $\mathbb{M}^{-1}(\mathbb{M}(x))$ contains one point, is dense in $\mathbb{R}^N$.

$\Diamond$

Here generic frames denote an open and dense set of frames, with respect to the topology induced by the Grassmanian manifold topology (see [2]).

In a completely different line of research, Donoho and Huo obtained in their seminal paper [3] an equivalence result for solving sparse optimization problems. More specifically let us consider the following objects.

$$T = \left[\begin{array}{cc} I & R \end{array}\right] \quad (1)$$

$$D(N) = \frac{1}{2}(1+\sqrt{N}) \quad (2)$$

where I is the $N\times N$ identity matrix, and R is the N-point unitary FFT matrix. Let $\|c\|_p=(\sum_{k=1}^M|c_k|^p)^{1/p}$ for any M-vector c, and $p>0$. For $p=0$, we define $\|c\|_0$ to be the number of nonzero entries of c. With these notations Donoho and Huo showed the following result:

Theorem 1.2: Assume $x=Tc'$ for some $c'\in\mathbb{C}^{2N}$ so that $\|c\|_0<D(N)$. Then the following two optimization problems admit the same unique solution c':

$$\hat{c}_0 = argmin_{c:x=Tc}\|c\|_0 \quad (3)$$

$$\hat{c}_1 = argmin_{c:x=Tc}\|c\|_1 \quad (4)$$

$\Diamond$

The bound $D(N)$ as stated here was next improved by Elad and Bruckstein in [5] to $D(N)=(\sqrt{2}-\frac{1}{2})\sqrt{N}$, and also extended to other matrices than this particular T (see also [4], [6], [7]).

Optimization problems of type (3) and (4) are also related to sparse multicomponent signal decompositions. More specifically, consider the following estimation problem. Given the model

$$x=Us+Vt=\left[\begin{array}{cc} U & V\end{array}\right]\left[\begin{array}{c} s\\ t\end{array}\right] \quad (5)$$

where $x\in\mathbb{R}^N$ is the vector of measurements, $s,t\in\mathbb{R}^M$ are vectors of unknown component coefficients, U,V are known $N\times M$ mixing matrices, the problem is to obtain the Maximum A Posteriori (MAP) estimator of the two components Us and Vt, when s and t are known to have prior distributions of the form

$$p_S(s)\propto exp(-\alpha\|s\|_p^p)\ \ ,\ \ p_T(t)\propto exp(-\beta\|t\|_p^p) \quad (6)$$

It is then immediate to derive the MAP estimator as:

$$(\hat{s},\hat{t})_{MAP}=argmin_{Us+Vt=x}\ \ \alpha\|s\|_p^p+\beta\|t\|_p^p \quad (7)$$

Note for $p<1$, prior distributions of type (6) have long tails, are peakier than Gaussian (p=2), or even Laplacian (p=1) distributions, and allocate uniformly larger costs for nonzero components compared to the vanishing components.

In this short note we present a necessary condition for perfect reconstruction from the absolute value of the frame coefficients in terms of a sparse representation optimization

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problem. Furthermore, the optimization problem gives the solution for the reconstruction problem.

II. Notations

We use the following notations.

Let $F = [f_1|\cdots|f_M]$ denote the $N \times M$ real matrix whose columns are the M frame vectors $\{f_1, \ldots, f_M\}$ in $\mathbb{R}^N$. Let $G = [\tilde{f}_1|\cdots|\tilde{f}_M]$ denote the $N \times M$ matrix whose columns are the canonical dual frame vectors. That is,

$$FG^T = GF^T = I_N \quad , \quad \text{(perfect reconstruction)}$$
$$F^TG = G^TF = P \quad , \quad \text{(projection onto the coefficients space)}$$

Recall the map we are interested in is:

$$\mathbb{M}: \ \mathbb{R}^N/\{+1,-1\} \to \mathbb{R}^M \quad , \quad \mathbb{M}(x) = \{|\langle x, f_k\rangle|\}_{1\le k\le M} \tag{8}$$

where $\mathbb{R}^N/\{+1,-1\} = \{\hat{x}\ ;\ \}$ is the set of classes of vectors $\hat{x}$ obtained by identification of x with $-x$, that is $\hat{x} = \{x, -x\}$. Perfect reconstruction up to a sign of the vector x is possible if and only if $\mathbb{M}$ is injective.

Let $a \in Ran\mathbb{M} := \{|F^Tx|\ ;\ x \in \mathbb{R}^N\}$. We denote by A, K, and α the $2M \times 2M$ real matrices, respectively the $2M$ vector, defined by

$$A = \begin{bmatrix} I & I \\ I-P & -(I-P) \end{bmatrix} \ , \ K = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \ , \ \alpha = \begin{bmatrix} a \\ 0 \end{bmatrix} \tag{9}$$

For vectors $u \in \mathbb{R}^m$, $u \ge 0$ means $u_k \ge 0$ for every k. For $y \in \mathbb{R}^m$, we denote $|y|$ the vector of absolute values of its entries, $|y| = (|y_k|)_{1\le k\le m}$. We also let $supp(y)$ denote the support of the vector y, that is $supp(y) = \{k\ ;\ y_k \ne 0\}$.

III. Main Result

Theorem 3.1: Let $a \in Ran\,\mathbb{M}$. If $\mathbb{M}^{-1}(a)$ contains only one point, say $\{\hat{x}\}$, then for every $0 \le p < 1$ the following optimization problem

$$\underset{Au=\alpha}{argmin} \ \|u\|_p \tag{10}$$

admits exactly two solutions u and $\tilde{u} = Ku \in \mathbb{R}^{2M}$ independent of p, with $u = [t^T\ s^T]^T$ so that $a = |t+s|$, and $\hat{x} = \{G(t-s), -G(t-s)\}$.

Conversely, if for some $0 \le p < 1$ the optimization problem (10) admits at most two solutions, then $\mathbb{M}^{-1}(a)$ contains exactly one point. $\diamondsuit$

The proof is based on two key ingredients: one is the decomposition of any real number into its positive and negative parts; the other ingredient is the inequality $|a+b|^p \le |a|^p + |b|^p$, for all $a, b \in \mathbb{R}$ and $0 \le p < 1$, where the equality holds if and only if one of a, b is zero.

Proof

(1) Proof of $\Rightarrow$.

Let $x \in \mathbb{M}^{-1}(a)$. Thus $a = |F^Tx|$. Let t, s be the positive, and respectively the negative part of F^Tx, that is:

$$t_k = \begin{cases} (F^Tx)_k & if \quad (F^Tx)_k \ge 0 \\ 0 & otherwise \end{cases} \tag{11}$$

$$s_k = \begin{cases} 0 & if \quad (F^Tx)_k \ge 0 \\ -(F^Tx)_k & otherwise \end{cases} \tag{12}$$

for all $1 \le k \le M$. Note $t, s \ge 0$. We claim $u = [t^T\ s^T]^T$ and $\tilde{u} = [s^T\ t^T]^T$ are the only solutions of (10).

Since they satisfy $F^Tx = t - s$, we obtain $a = |t+s|$, and $x = G(t-s)$.

To prove the claim, first note that both u and $\tilde{u}$ are feasible vectors, that is they satisfy the linear constraint $Au = \alpha$. Let $u' = [v^T\ w^T]^T$ be another feasible vector, that is $Au' = \alpha$. We will prove $\|u'\|_p \ge \|u\|_p$, and $\|u'\|_p = \|u\|_p$ if and only if either $u' = u$, or $u' = \tilde{u}$.

Indeed, we have $u' = u + d$, where $d \in ker\,A$. Due to the special form of A in (9), we obtain:

$$d = \begin{bmatrix} h \\ -h \end{bmatrix} \quad , \quad h \in Ran\,P \tag{13}$$

Let $J = supp(d) = \{k\ ;\ d_k \ne 0\} \subset \{1, 2, \ldots, 2M\}$, $I = supp(u)$, and $I_0 = I \setminus J$, $I_1 = I \cap J$, and $I_2 = J \setminus I$. Then $supp(u') \subset I_0 \cup I_1 \cup I_2$ and

$$\|u'\|_p^p = \sum_{k\in I_0} |u_k|^p + \sum_{k\in I_1} |u_k + d_k|^p + \sum_{k\in I_2} |d_k|^p \tag{14}$$

Let $\sigma : \{1, 2, \ldots, 2M\} \to \{1, 2, \ldots, 2M\}$ be the map $\sigma(m) = m + M$, if $m \le M$, and $\sigma(m) = m - M$ for $m > M$. Let $j : \{1, 2, \ldots, 2M\} \to \{1, 2, \ldots, M\}$ be the map $j(m) = m$, for $m \le M$, and $j(m) = m - M$, for $m > M$. Because of special form of d, $k_0 \in J$ if and only if $\sigma(k_0) \in J$, that is $\sigma(J) = J$. On the other hand, the choice of t and s guarantees that $k_0 \in I$ if and only if $\sigma(k_0) \notin I$, that is $I \cap \sigma(I) = \emptyset$. Thus if $k \in I_1$ then $\sigma(k) \in I_2$ (although $\sigma(I_1) \subset I_2$ may be strict) and $|d_k| = |d_{\sigma(k)}| = |h_{j(m)}|$. Let $I_{21} = \sigma(I_1)$ and $I_{22} = I_2 \setminus I_{21}$. Thus $I_2 = I_{21} \cup I_{22}$, and

$$\begin{aligned} \|u'\|_p^p &= \sum_{k\in I_0} |u_k|^p + \sum_{k\in I_1} |u_k + d_k|^p + \sum_{k\in I_{21}} |d_k|^p + \sum_{k\in I_{22}} |d_k|^p \\ &= \|u\|_p^p + \sum_{k\in I_1} [|u_k + d_k|^p + |d_k|^p - |u_k|^p] + \sum_{k\in I_{22}} |d_k|^p \end{aligned} \tag{15}$$

For $0 \le p < 1$, we have $|u_k + d_k|^p + |d_k|^p \ge |u_k|^p$ with equality achieved if and only if either $d_k = 0$, or $d_k = -u_k$. This proves that:

$$\|u'\|_p \ge \|u\|_p \tag{16}$$

and thus u is a global optimizer for (10).

The only remaining issue is to prove that if $\|u'\|_p = \|u\|_p$, then either $u' = u$, or $u' = Ku$. Equations (15) prove that $\|u'\|_p = \|u\|_p$ if and only if $I_{22} = \emptyset$ and for all $k \in I_1$, either $d_k = -u_k$, or $d_k = 0$. However $d_k \ne 0$ for $k \in I_1$ (by construction), hence $d_k = -u_k$ for all $k \in I_1$. This means:

$$(u')_k = \begin{cases} u_k & for \quad k \in I_0 \\ u_k & for \quad k \in I_{21} = \sigma(I_1) \\ 0 & otherwise \end{cases} \tag{17}$$

Let t', s' be the M-components of u', $u' = [(t')^T\ (s')^T]^T$. The feasibility constraint $Au' = \alpha$ implies that $t' + s' = a$, and $(I-P)(t'-s') = 0$. Let $c' = t' - s'$. Thus $Pc' = c'$, hence $c' \in Ran\,F^T$. On the other hand, since $I_{22} = \emptyset$ and $\sigma(I_1) \subset I_2$, we have that $\sigma(I_1) = I_2$. Hence, $\sigma(I_2) = \sigma^2(I_1) = I_1$ and so $I_0 \cap \sigma(I_2) = I_0 \cap I_1 = \emptyset$. Also, by construction $I_0 \cap I_2 = \emptyset$ and it follows $\sigma(supp(u')) \cap supp(u') = \emptyset$. Thus $|c'| = t' + s' = a$. This means $x' = Gc'$ is in $\mathbb{M}^{-1}(a)$. Since $\mathbb{M}^{-1}(a)$ contains only one point, $\hat{x} = \{x, -x\}$, it follows that either $u' = u$, or $u' = -u$, which ends the proof of this way.

(2) Proof of $\Leftarrow$.

Fix $a \in \mathbb{M}$, and assume $\mathbb{M}^{-1}$ has at least two classes, say $\hat{x} = \{x, -x\}$ and $\hat{x'} = \{x', -x'\}$. Consider again the positive and negative parts of $F^T x$ and $F^T x'$, respectively, say t, s, and t', s'. Since $x \neq x'$ and $x \neq -x'$, yet $|F^T x| = |F^T x'|$, it follows that $t' \neq t$ and $t' \neq s$, yet $\|u\|_p = \|u'\|_p$, where $u = [t^T \ s^T]^T$ and $u' = [t'^T \ s'^T]^T$. Thus u, Ku, u', Ku' are distinct solutions of (10), which proves the converse.

Q.E.D. $\diamondsuit$

Remark 1: Note that for $p = 1$ the result does not hold true. Indeed, for $p = 1$ any feasible vector u of (10) of positive components has the same l^1 norm: $\|u\|_1 = \|a\|_1$.

Remark 2: If u is a feasible vector, that is it satisfies $Au = \alpha$, then Ku is also a feasible vector, that is it satisfies $AKu = \alpha$, In particular, if u is a solution of (10) then Ku is also a solution of same problem.

Remark 3: Note that we do not impose any positivity constraint on u in (10). And yet, remarkably, the optimizer turns out to have nonnegative components.

Remark 4: Connections between ℓ^p optimization problems and sparse signal representations have been studied in literature in the context of solving ICA type problems; see [8], [9], [1].

Corollary 3.2: If $\mathbb{M}$ is injective, then for all $a \in Ran\,\mathbb{M}$, and $0 \leq p < 1$, the optimization problem (10) admits only two solutions u and Ku so that $\mathbb{M}^{-1}(a) = \{G(t-s), G(s-t)\}$, where $u = [t^T \ s^T]^T$.

Corollary 3.3: If (10) admits at most one solution for all a, then $\mathbb{M}$ is injective.

IV. Conclusions

In this paper we study the reconstruction problem of a real signal when only absolute values of its real frame coefficients are known. In general one can expect at most to reconstruct the original signal up to an ambiguity of one global sign. We prove that this is the case if and only if an ℓ^p optimization problem, more specifically (10), admits exactly two solutions. Furthermore, the solutions of this problem are directly related to the original (and reconstructed) signal. This result reduces a combinatorial optimization problem (where the combinatorics are due to the exhaustive search over all possible sign combinations) to an ℓ^p optimization problem (albeit nonconvex).

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