Measure and Redundancy of Frames

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ABSTRACT

In this paper we present an abstract theory of frame redundancy. More specifically we define the frame measure function as a representation of the set of frames indexed by the same index set, in the set of continuous real-valued functions over a compact space, that is compatible with a special partial ordering introduced in this paper, it is normalized and it is additive with respect to orthogonal superframes. A frame measure function is as relevant as the equivalence and partial ordering relations are. Thus we will spend some time in trying to convince the reader of the relevance of our new proposed equivalence relation. Here we are going to present only basic properties of these concepts.

Keywords: frame, frame measure, redundancy, equivalence of frames

1. INTRODUCTION

The notion of frame has been introduced by Duffin and Schaefer in their seminal paper.[13] Since then many studies were published and the theory enlarged and advanced so today it covers a respectable number of fields all over mathematics, engineering and physiscs. In this context, we refer the reader to the expository papers,[12] ,[21] or more recently.[8]

For a Hilbert space H, a *(Hilbert) frame* indexed by a countable index set I is a set of vectors $\mathcal{F} = \{f_i\ ;\ i \in I\}$ of H such that there are two positive numbers $A, B > 0$ so that for any vector $x \in H$,

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, f_i\rangle|^2 \leq B\|x\|^2 \tag{1}$$

Directly from the definition one can see that $\mathcal{F}$ is necessary a complete system in H (that is, its linear span is dense in H). Indeed, if x is a vector orthogonal to the closure of the span of $\mathcal{F}$, then the sum in (1) vanishes, which makes the norm of x zero, and then x vanishes as well. In most of the cases $\mathcal{F}$ is an overcomplete set, meaning that one can remove some subset and leave the remaining set complete, or even frame (,[56]). The only case when this cannot happen is when $\mathcal{F}$ is a Riesz basis. A *Riesz basis* is a complete set $\mathcal{F} = \{f_i\ ;\ i \in I\}$ so that for two numbers $A, B > 0$ and every finite sequence of complex numbers $c = (c_i)_{i\in I}$,

$$A\sum_{i\in I} |c_i|^2 \leq \|\sum_{i\in I} c_i f_i\|^2 \leq B\sum_{i\in I} |c_i|^2 \tag{2}$$

It turns out a Riesz basis is also a frame and the constants A, B in (1) can be chosen as in (2). When the set $\mathcal{F}$ satisfies only (2) but is not necessarily complete, we say $\mathcal{F}$ is a *Riesz basis for its span*, or a *Riesz basic sequence*, or a *s-Riesz basis*. Similarly, $\mathcal{F}$ is a *frame for its span* when (1) holds true for all x in the closure of the linear span of $\mathcal{F}$. The frame property represents a special case of completeness, whereas Riesz basis for its span corresponds to a special case of linear independence. Riesz basis property represents the intersection of both frame and Riesz basis for its span properties.

All these facts are well-known in literature, and many applications (in signal processing, for instance) were presented. Both the theory and applications states that frames are in general redundant sets, and therefore vectors of the Hilbert space are not uniquely decomposed in terms of the frame set vectors. However there

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was no attempt to quantify how "redundant" a frame set is. The purpose of this paper is to present a formal theory of frame redundancy and connect it to some areas of operator theory, topology, information theory, signal processing, and representation theory.

To fix the notations, in this paper we consider I a countable index set, and frames are always sets of vectors of some Hilbert space (not necessarily the same) indexed by I, where the indexing *does matter*. Because of tradition we write $\mathcal{F} = \{f_i \; ; \; i \in\}$, however by a frame we will always mean a map $f : I \to H$, so that $\mathcal{F} = \{f_i = f(i) \; ; \; i \in I\}$ satisfies (1) for some $A, B > 0$. The set of all frames index by I is denoted $\mathcal{F}[I]$. The numbers A, B in (1) are called *frame bounds*. A frame is called *tight* if we can choose equal frame bounds. When the frame bounds can be set to one, $\mathcal{F}$ is called *Parseval frame* (or normalized tight frame).

For a frame $\mathcal{F} = \{f_i; i \in I\}$ for H, we define the following operators: the *analysis map* is the bounded linear map $T : H \to l^2(I)$ defined by $T(x) = \{\langle x, f_i\rangle\}_{i\in I}$; the *synthesis map* is the adjoint of T, $T^* : l^2(I) \to H$, $T^*(c) = \sum_{i\in I} c_i f_i$; the *frame operator* defined by $S : H \to H$, $S = T^*T$, $x \mapsto S(x) = \sum_{i\in I}\langle x, f_i\rangle f_i$; the *Gramm operator* defined as $G : l^2(I) \to l^2(I)$, $G = TT^*$, $c \mapsto (G(c))_i = \sum_{j\in I}\langle f_j, f_i\rangle c_j$. The *canonical dual* frame of $\mathcal{F}$ is the frame $\tilde{\mathcal{F}} = \{\tilde{f}_i \; ; \; i \in I\}$ defined by $\tilde{f}_i = S^{-1} f_i$. In general, we call a frame $\mathcal{G} = \{g_i \; ; \; i \in I\}$ in H *dual* of $\mathcal{F}$ if for any $x \in H$,

$$\sum_{i\in I}\langle x, f_i\rangle g_i = \sum_{i\in I}\langle x, g_i\rangle f_i = x \tag{3}$$

In particular, the canonical dual frame is a dual frame. Equation (3) says a pair of frame and one of its duals gives a discrete resolution of the identity of the Hilbert space where the frame lives. The canonical dual frame gives more information about the frame than just a discrete resolution of identity. Let us denote by E the range of T in $l^2(I)$. We call E the *range of coefficients*. The lower frame bound guarantees that E is closed. Let us denote by P the orthogonal projection onto E in $l^2(I)$. In the canonical basis $(\delta_i)_{i\in I}$ of $l^2(I)$, $\delta_i = (\delta_{i,j})_{j\in I}$, where $\delta_{i,j}$ is the Kronecker symbol (1 for $i = j$, and 0 otherwise), the projection P has the following representation (whose validity is easily verifiable by the reader):

$$P_{i,j} := \langle P\delta_i, \delta_j\rangle = \langle f_i, \tilde{f}_j\rangle = \langle \tilde{f}_i, f_j\rangle \tag{4}$$

Note if $\mathcal{F}$ is a Riesz basis, then the range of T is the whole space $l^2(I)$, the projection P becomes the identity of $l^2(I)$, and the canonical dual has the biorthogonal property $\langle f_i, \tilde{f}_j\rangle = \delta_{i,j}$. In this case (and only in this case) the canonical dual is the unique dual of $\mathcal{F}$. In general there may be more dual dual frames. The operator in $l^2(I)$ whose matrix elements in the canonical basis $(\delta_i)_{i\in I}$ are the inner product between the frame vectors and the dual frame vectors, is, in this case, a nonorthogonal projection. The only case when this projection is orthogonal is when the dual frame is the canonical dual frame.

Another object constructed from a frame is the *associated Parseval frame* $\mathcal{F}^\# = \{f_i^\# \; ; \; i \in I\}$ defined by $f_i^\# = S^{-1/2} f_i$. Indeed, one can easily prove that $\mathcal{F}^\#$ is a Parseval frame. The coefficient ranges of the canonical dual frame and associated Parseval frame coincide with the range of coefficients of $\mathcal{F}$. Note the entries of P in the canonical basis $(\delta_i)_{i\in I}$ can be computed also through

$$P_{i,j} = \langle f_i^\#, f_j^\#\rangle \tag{5}$$

In particular the diagonal elements are given by

$$P_{i,i} = \|f_i^\#\|^2 = \langle f_i, \tilde{f}_i\rangle \tag{6}$$

and are real between 0 and 1.

Let us describe now the approach. Our approach is inspired by the Cantor cardinal numbers theory (which, in turn, owes it to Gottlob Frege; see for instance[17]). The cardinal number is defined as the class of equivalent sets, where two sets are equivalent if there is a bijection mapping one into the other. Moreover, injective and surjective maps define order relations on equivalent classes of sets. A similar approach was taken by Murray and von Neumann in their comparison of projection theory. Two projections in a von Neumann algebra are equivalent if there is a partial isometry of the algebra that maps the range of one projection onto the range of the other projection. Then, in a finite algebra, equivalent classes of projections are characterized uniquely by the center valued trace. With these two models in mind, our approach is as follows: first we have to define equivalent frames, and consider the class of equivalent frames; at the same time we have to introduce a partial ordering compatible with the equivalence relation, in order to compare (some) classes of frames. Once this is done, the actual comparison is performed through a representation theory model. More specifically we define the frame measure function as a representation of $\mathcal{F}[I]$ in the set of continuous real-valued functions over a compact space, that is compatible with the partial ordering introduced before, is normalized and satisfies a special additivity property. A frame measure function is as relevant as the equivalence and partial ordering relations are. Thus we will spend some time in trying to convince the reader of the relevance of our new proposed equivalence relation. The rational for our definition comes from both analytic and practical reasons. We try to extend the clear meaning of redundancy in the finite dimensional case to the infinite dimensional setting. If the frame set has M vectors in the N-dimensional complex space $\mathbf{C}^N$, then the redundancy is defined as $\frac{M}{N}$. Then what we need to do is to find the relevant transformation properties if the frame sets that leave this ratio constant. As we will argue later, finite permutations and arbitrary change of phase are among the relevant transformations. Also an information theory argument suggests how to define equivalent classes when the previous ratio is properly connected to the trace of the projection P. All these is done in Section 2. Next, the frame measure function is defined in Section 3, as mentioned before. The rest of the section, is spent on proving several properties of these frame measure functions.

A similar theory can be made to measure Riesz bases for their span. A future paper will address this issue.

2. FRAME ORDERING

In this section we revised the commonly known frame equivalence relation and argue about some limitations this relation has. To address those problems, we introduce a new equivalence relation that will constitute the foundation of our comparison theory.

2.1. The standard equivalence relation

Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ be two frames for, respectively H_1,H_2. Then

DEFINITION 2.1. *We say $\mathcal{F}_1 \sim \mathcal{F}_2$ if there is a bounded invertible operator $S : H_1 \to H_2$ such that $Sf_i^1 = f_i^2$ for every $i \in I$.* This is an equivalence relation as can be easily proved (namely it is reflexive, symmetric and transitive). Moreover, it admits the following geometric interpretation:

Theorem 2.2 ([3,11]). *Consider $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ and P_1, P_2 their associated orthogonal projections onto the coefficients range. Then $\mathcal{F}_1 \sim \mathcal{F}_2$ if and only if $P_1 = P_2$.*

What this result says is that two frames are equivalent $\sim$ if and only if they span the same range of coefficients. Notice that a frame functional calculus associated to some frame $\mathcal{F} \in \mathcal{F}[I]$ would always produce frames equivalent to $\mathcal{F}$. By frame functional calculus we mean frames of the form $\mathcal{G} = \{g_i; g_i = \phi(S)f_i\}$ for some Borel measurable function ϕ so that $\phi([A, B]) \subset [C, D] \subset (0, \infty)$, with A, B the frame bounds of $\mathcal{F}$. In particular, the canonical dual frame (obtained for $\phi(x) = 1/x$) is $\sim$ equivalent to the original frame $\mathcal{F}$. By the previous theorem, all the frames obtained through functional calculus from $\mathcal{F}$ span the same space of coefficients in $l^2(I)$.

For our purposes, this equivalence relation is not sufficient. In other words, we need to enlarge the class of equivalent frames beyond the rigid constraint Theorem 2.2 poses on the range of coefficients.

The following remarks present two properties we would like equivalent frames to have, but the $\sim$ equivalence relation fails to have.

REMARK 2.3. *Let $\mathcal{F} \in \mathcal{F}[I]$. Note that by arbitrarily changing the sign of each vector we obtain $\sim$ nonequivalent frames. However we would like the new frame sets thus obtained to remain $\sim$ equivalent to $\mathcal{F}$. More generally, for an arbitrary set of phases $(\phi_i)_{i\in I}$ construct $\mathcal{G} = \{g_j = e^{i\phi_j} f_j; j \in I\}$. Note $\mathcal{G}$'s remain frame, but in general they are no longer $\sim$ equivalent to $\mathcal{F}$. We would like the new equivalence class of $\mathcal{F}$ to contain all the $\mathcal{G}$'s obtained this way.*

REMARK 2.4. *Another operation we would like the class of equivalent frames to possess is finite permutation. In general, for $\mathcal{F} \in \mathcal{F}[I]$ and $\pi : I \to I$ a finite permutation, $gc = \{g_i = f_{\pi(i)}; i \in I\}$ is no longer $\sim$ equivalent to $\mathcal{F}$. We will require the new equivalency class to be invariant to this kind of transformation.*

Beside these two classes of transformations the new equivalence classes have to be invariant to, there is another criterion in comparing two frames that we want to incorporate in the new theory. This is furnished by a stochastic signal analysis that generalizes the arguments presented in.[12]

First note that there is no lack of generality by restricting the analysis to normalized tight frames since any frame is $\sim$ equivalent to its associated normalized tight frame, and by transitivity, once a new equivalence relation is introduced on normalized tight frames, it immediately extends to arbitrary frames. Consider $\mathcal{F} \in \mathcal{F}[I]$ a normalized tight frame. Assume the span H of $\mathcal{F}$ models a class of signals we are interested to transmit using an encoding and decoding scheme based on $\mathcal{F}$ as in Figure 1. More specific, a "signal" (that is a vector) $x \in H$ is "encoded" through the sequence of coefficients $c = \{\langle x, f_i \rangle\}_{i \in I}$ given by the analysis operator $T : H \to l^2(I)$. These coefficients are sent through a communication channel to a receiver and there they are "decoded" using a linear reconstruction scheme $\hat{x} = \sum_{i \in I} d_i f_i$ furnished by the reconstruction operator T^*. Often it happens the transmitted coefficients $c = (c_i)_{i \in I}$ are perturbed by some (channel) noise. Hence the received coefficients $d \neq c$. We assume an *additive white noise channel* model, meaning the transmitted coefficients are perturbed additively by unit variance white noise, that is

$$d_i = c_i + n_i \tag{7}$$
$$\mathbf{E}[n_i] = 0 \tag{8}$$
$$\mathbf{E}[n_i \overline{n_j}] = \delta_{i,j} \tag{9}$$

where $\mathbf{E}$ is the expectation operator. Then the reconstructed signal $\hat{x}$ has two components, one due to the transmitted coefficients $\sum_i c_i f_i = x$ and the other due to the noise $\varepsilon = \sum_i n_i f_i$. We analyse the noise due component ε. Since its variance is infinite in general (this, in turn, implies some convergence problem in defining $\sum_i n_i f_i$!), we consider the case that only finitely many coefficients are transmitted, say a finite subset $I_n \subset I$. Then the *average variance per coefficient of the noise-due-error* is defined by:

$$a_n = \frac{\mathbf{E}[|e_n|^2]}{|I_n|} \tag{10}$$

where

$$e_n = \sum_{i \in I_n} n_i f_i \tag{11}$$

Using the assumptions (8),(9) we obtain

$$a_n = \frac{1}{|I_n|} \sum_{i \in I_n} \|f_i\|^2 \tag{12}$$

Since $\|f_i\| \leq 1$ it follows $a_n \leq 1$. Note that if instead of $\mathcal{F}$ an orthonormal basis was used, the average noise due error variance per coefficient would have been

$$b_n = 1 \tag{13}$$

Hence a_n gives a measure of how much the channel noise variance is reduced when a frame is used instead of an orthonormal basis. In channel encoding theory, the noise reduction phenomenon described before is attributed to the redundancy a frame has compared to an orthonormal basis (see for instance[12]). Hence, any measure of redundancy has to be connected to these averages a_n from (12).

We end this subsection with a few comments on topology of frames. The correspondence between $\sim$ equivalence and orthogonal projections in $B(l^2(I))$ induces a quasi-metric on $\mathcal{F}[I]$ (respectively a metric on $\mathcal{F}[I]/\sim$):

$$d(\mathcal{F}_1, \mathcal{F}_2) = \|P_1 - P_2\| \quad , \quad \mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I] \tag{14}$$

where $P_1 = \tilde{T}_1 T_1^*$, $P_2 = \tilde{T}_2 T_2^*$ are the orthogonal projectors onto the coefficient spans associated to the two frames. In turn this quasi-metric defines a topology on $\mathcal{F}[I]$ denoted by τ. A basis of open sets in τ is given by $\{\mathcal{G} \in \mathcal{F}[I] \mid d(\mathcal{G}, \mathcal{F}) < \varepsilon\}$ for every $\varepsilon > 0$ and $\mathcal{F} \in \mathcal{F}[I]$. Thus a map $F : \mathcal{F}[I] \to X$ from $\mathcal{F}[I]$ to a topological space (X, Σ) is continuous at $\mathcal{F} \in \mathcal{F}[I]$ if and only if for any neighborhood $U \in \Sigma$ of $F(\mathcal{F})$, there is an $\varepsilon > 0$ so that for any $\mathcal{G} \in \mathcal{F}[I]$ with $d(\mathcal{F}, \mathcal{G}) < \varepsilon$ then $F(\mathcal{G}) \in U$.

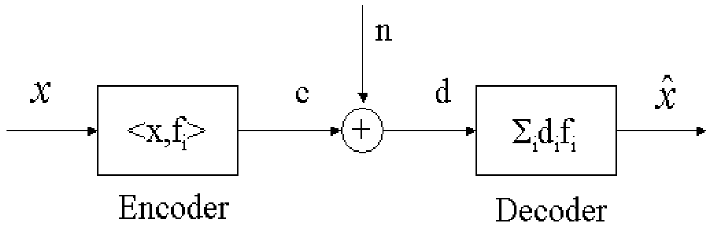

Figure 1. The Transmission Encoding-Decoding Scheme used to suggest the importance of averages (12).

2.2. The New Equivalence Relation

Based on the remarks and discussion presented in the previous subsection, we define a new equivalence relation as follows. First fix a sequence of covering, nested and finite subsets of I, $(I_n)_{n\geq 0}$, that is

$$I_0 \subset I_1 \subset \cdots \subset \quad I_n \quad \subset I_{n+1} \subset \cdots \subset I \tag{15}$$

$$|I_n| \quad < \quad \infty \tag{16}$$

$$\cup_{n\geq 0} I_n \quad = \quad I \tag{17}$$

DEFINITION 2.5. *Two frames $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ are said* $\approx$ equivalent with respect to $(I_n)_{n\geq 0}$ *if*

$$\lim_{n\to\infty} \frac{1}{|I_n|} \sum_{i\in I_n} (\langle f_i^1, \tilde{f}_i^1\rangle - \langle f_i^2, \tilde{f}_i^2\rangle) = 0 \tag{18}$$

From this point on, we assume the same sequence $(I_n)_{n\geq 0}$ of subsets of I, unless otherwise indicated. To simplify the notation, we shall drop the explicit mention of the sequence $(I_n)_{n\geq 0}$ in $\approx$ equivalence. Thus we shall simply say $\mathcal{F}_1$ and $\mathcal{F}_2$ are $\approx$ equivalent, or $\mathcal{F}_1 \approx \mathcal{F}_2$, if (18) holds true.

It is a simple exercise to show that $\approx$ is a veritable equivalency relation, that is $\mathcal{F} \approx \mathcal{F}$ (reflexivity), $\mathcal{F}_1 \approx \mathcal{F}_2 \Rightarrow \mathcal{F}_2 \approx \mathcal{F}_1$ (symmetry) , and $\mathcal{F}_1 \approx \mathcal{F}_2 \wedge \mathcal{F}_2 \approx \mathcal{F}_3 \Rightarrow \mathcal{F}_1 \approx \mathcal{F}_3$ (transitivity).

Next, note this equivalence relation solves the "desired" properties mentioned in the previous section. Firstly, if $\mathcal{F}_1 \sim \mathcal{F}_2$ then $P_1 = P_2$ (by Theorem 2.2), hence $\langle f_i^1, \tilde{f}_i^1\rangle = \langle f_i^2, \tilde{f}_i^2\rangle$, for every $i \in I$, hence (18) holds true, hence $\mathcal{F}_1 \approx \mathcal{F}_1$. Secondly, $\langle f_i, \tilde{f}_i\rangle$ is invariant to an arbitrary change of phase of vectors f_i, since $\tilde{f}_i = S^{-1} f_i$ would change by the same phase. Thus $\approx$ equivalency classes are invariant to arbitrary change of phase. Thirdly, for a finite permutation, the average $\frac{1}{|I_n|}\sum_{i\in I_n}\langle f_i, \tilde{f}_i\rangle$ remains the same for $n \geq N$, for sufficiently large N. Hence $\approx$ equivalency classes are invariant to finite permutations of the frame sets. Finally, condition (18) requires the averages of the type (12) associated to the two frames $\mathcal{F}_1$, $\mathcal{F}_2$ should behave the same. The following result whose proof is immediate makes more precise this statement:

Proposition 2.6. *$\mathcal{F}_1 \approx \mathcal{F}_2$ if and only if, for any sequence $(n_k)_{k\geq 0}$ so that $\gamma := \lim_{k\to\infty} \frac{1}{|I_{n_k}|}\sum_{i\in I_{n_k}}\langle f_i^1, \tilde{f}_i^1\rangle$ exists, the corresponding limit of averages associated to $\mathcal{F}_2$ exists as well and equals γ, that is*

$$\lim_{k\to\infty} \frac{1}{|I_{n_k}|} \sum_{i\in I_{n_k}} \langle f_i^2, \tilde{f}_i^2\rangle = \gamma \tag{19}$$

To simplify the notation, we denote by $a_n(\mathcal{F})$ the average:

$$a_n(\mathcal{F}) = \frac{1}{|I_n|} \sum_{i\in I_n} \langle f_i, \tilde{f}_i\rangle \tag{20}$$

Since for any frame $0 \leq \langle f_i, \tilde{f}_i \rangle \leq 1$, it follows $0 \leq a_n(\mathcal{F}) \leq 1$, for every n.

The following result shows that every $\approx$ equivalence class contains frames of a special form, namely frames made of union of an orthonormal basis with a number of zero vectors. It is important to note that the indexing and particular choice of $(I_n)_{n\geq 0}$ does matter. Later on we shall see conditions on the nested sequence $(I_n)_{n\geq 0}$ to produce the same classes of equivalence.

Theorem 2.7. *Let $\mathcal{F} \in \mathcal{F}[I]$ be a frame index by I. Then there are two frames $\mathcal{G}_1 = \{g_i^1; i \in I\}$, $\mathcal{G}_2 = \{g_i^2; i \in I\}$ both $\approx$ equivalent to $\mathcal{F}$ with the following properties:*

1. *For every i, g_i^1 is either zero, or g_i^1 belongs to an orthonormal set B, so that for any $i \neq j$ if $g_i^1 \neq 0$, $g_j^1 \neq 0$, then $g_i^1 \neq g_j^1$;*
2. *For every i, g_i^2 is either zero, or g_i^2 belongs to the orthonormal set B, so that for any $i \neq j$ if $g_i^2 \neq 0$, $g_j^2 \neq 0$, then $g_i^2 \neq g_j^2$;*
3. *$\|g_i^1\| \leq \|g_i^2\|$, that is, when $g_i^1 \neq 0$, then necessarily $g_i^2 \neq 0$;*
4. *The following inequalities hold true:*

$$a_n(\mathcal{G}_1) \leq \quad a_n(\mathcal{F}) \quad \leq a_n(\mathcal{G}_2) \tag{21}$$

$$a_n(\mathcal{F}) - a_n(\mathcal{G}_1) \quad \leq \quad \frac{1}{|I_n|} \tag{22}$$

$$a_n(\mathcal{G}_2) - a_n(\mathcal{F}) \quad \leq \quad \frac{1}{|I_n|} \tag{23}$$

The proof of this result is fairly simple, but since we will use this result many times in the following, we include it here.

Proof

The proof is constructive and is as follows. Assume an enumeration of I $k \in \mathbf{N} \mapsto i_k \in I$ is fixed so that $|I_n| < k \leq |I_{n+1}|$ for $i_k \in I_{n+1} \setminus I_n$. Assume $B = \{e_k; k \geq 1\}$ is an orthonormal set. We constructive inductively $\mathcal{G}_1, \mathcal{G}_2$. Let $\lfloor x \rfloor$, be the largest integer smaller than or equal to x, and $\lceil x \rceil$ be the smallest integer larger than or equal that x. Initialization. Assume N is the first integer so that $I_N \neq \emptyset$. Set $\mathcal{G}_1^k = \emptyset$ and $\mathcal{G}_2^k = \emptyset$, for $0 \leq k < N$. Set $s_-^N = \lfloor \sum_{i \in I_N} \langle f_i, \tilde{f}_i \rangle \rfloor$, $s_+^N = \lceil \sum_{i \in I_N} \langle f_i, \tilde{f}_i \rangle \rceil$. Hence $\frac{1}{|I_N|} s_-^N \leq a_N(\mathcal{F}) \leq \frac{1}{|I_N|} s_+^N$. Set $\mathcal{G}_1^N = \{g_i^1; i \in I_N\}$, where $g_{i_k}^1 = e_k$ for $1 \leq k \leq s_-^N$, and $g_{i_k}^1 = 0$ for $s_-^N + 1 \leq k \leq |I_N|$. Similarly, set $\mathcal{G}_2^N = \{g_i^2; i \in I_N\}$, where $g_{i_k}^2 = e_k$ for $1 \leq k \leq s_+^N$, and $g_{i_k}^2 = 0$ for $s_+^N + 1 \leq k \leq |I_N|$.

Assume $\mathcal{G}_1^n$ and $\mathcal{G}_2^n$ are defined, for some $n \geq N$. Set $s_-^{n+1} = \lfloor \sum_{i \in I_{n+1}} \langle f_i, \tilde{f}_i \rangle \rfloor$, $s_+^{n+1} = \lceil \sum_{i \in I_{n+1}} \langle f_i, \tilde{f}_i \rangle \rceil$. Then:

$$\frac{1}{|I_{n+1}|} s_-^{n+1} \leq \quad a_{n+1}(\mathcal{F}) \quad \leq \frac{1}{|I_{n+1}|} s_+^{n+1} \tag{24}$$

$$a_{n+1}(\mathcal{F}) - \frac{1}{|I_{n+1}|} s_-^{n+1} \quad \leq \quad \frac{1}{|I_{n+1}|} \tag{25}$$

$$\frac{1}{|I_{n+1}|} s_+^{n+1} - a_{n+1}(\mathcal{F}) \quad \leq \quad \frac{1}{|I_{n+1}|} \tag{26}$$

Construct $\mathcal{G}_1^{n+1} = \{g_i^1; i \in I_{n+1} \setminus I_n\}$ and $\mathcal{G}_2^{n+1} = \{g_i^2; i \in I_{n+1} \setminus I_n\}$ as follows: $g_{i_k}^1 = e_k$ for $s_-^n < k \leq s_-^{n+1}$, $g_{i_k}^1 = 0$ for $s_-^{n+1} + 1 \leq k \leq |I_{n+1}|$; $g_{i_k}^2 = e_k$ for $s_+^n < k \leq s_+^{n+1}$, $g_{i_k}^2 = 0$ for $s_+^{n+1} + 1 \leq k \leq |I_{n+1}|$.

Now set $\mathcal{G}_1 = \cup_{n \geq 0} \mathcal{G}_1^n$, and $\mathcal{G}_2 = \cup_{n \geq 0} \mathcal{G}_2^n$.

We claim $\mathcal{G}_1, \mathcal{G}_2$ satisfy the conclusions of the statement. Indeed, each of the two sets consists of an orthonormal basis B and a number of zero vectors. Hence both are normalized tight frames for $H = \overline{span(B)}$. Since $s_-^n \leq s_+^n$ it follows $\|g_i^1\| \leq \|g_i^2\|$. Next, (21-23) follow from (24-26) by noting that $a_n(\mathcal{G}_1) = \frac{1}{|I_n|} s_-^n$ and $a_n(\mathcal{G}_2) = \frac{1}{|I_n|} s_+^n$.

□

REMARK 2.8. *The two frames $\mathcal{G}_1$, $\mathcal{G}_2$ constructed above have the following ordering property. Let E_1, E_2 denote the range of their associated analysis maps $T_k : H \to l^2(I)$, $T_k(x) = \{\langle x, g_i^k \rangle\}_{i\in I}$, $k = 1, 2$. Then $E_1 \subset E_2$, or, in the terminology that will be introduced in the next section, $\mathcal{G}_1 \leq \mathcal{G}_2$.*

Sequences $(I_n)_{n\geq 0}$ that yield the same classes of equivalence are characterized by the following sufficient condition. For the sake of simpler notation, assume $(I_n)_{n\geq 0}$ and $(J_n)_{n\geq 0}$ are extended by one term, namely $I_{-1} = J_{-1} = \emptyset$.

Theorem 2.9. *Suppose $(I_n)_{n\geq 0}$ and $(J_n)_{n\geq 0}$ are two covering, nested and finite sequences of subsets of I. Denote by $N_1(n)$ the largest integer so that $J_{N_1(n)} \subset I_n$, and by $N_2(n)$ the smallest integer so that $I_n \subset J_{N_2(n)}$. If*

$$\lim_{n\to\infty} \frac{|J_{N_1(n)}| - |I_n|}{|I_n|} = 0 \tag{27}$$

$$\lim_{n\to\infty} \frac{|J_{N_2(n)}| - |I_n|}{|I_n|} = 0 \tag{28}$$

then for every $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$, $\mathcal{F}_1 \approx \mathcal{F}_2$ with respect to $(I_n)_{n\geq 0}$ if and only if $\mathcal{F}_1 \approx \mathcal{F}_2$ with respect to $(J_n)_{n\geq 0}$;

Proof

Assume (27,28) hold true. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ be two equivalent frames with respect to $(J_n)_{n\geq 0}$. Then

$$\lim_{n\to\infty} \frac{1}{|J_n|} \sum_{i\in J_n} (\langle f_i^1, \tilde{f_i^1} \rangle - \langle f_i^2, \tilde{f_i^2} \rangle) = 0$$

Now

$$\frac{1}{|I_n|} \sum_{i\in I_n} (\cdot) = \frac{|J_{N_1(n)}|}{|I_n|} \frac{1}{|J_{N_1(n)}|} \sum_{i\in J_{N_1(n)}} (\cdot) + \frac{1}{|I_n|} \sum_{i\in I_n \setminus J_{N_1(n)}} (\cdot)$$

where $(\cdot) = \langle f_i^1, \tilde{f_i^1} \rangle - \langle f_i^2, \tilde{f_i^2} \rangle$. We compute separately the limits of the two right-side terms. The first term converges to zero because $\lim_{n\to\infty} |J_{N_1(n)}|/|I_n| = 1$ (by 27) and the previous relation. The second term converges to zero as well since $|\sum_{i\in I_n \setminus J_{N_1(n)}} (\cdot)| \leq |I_n| - |J_{N_1(n)}|$ and then apply (27).

For the converse, note that (27,28) are symmetric with respect to (I_n), (J_n). Indeed, let $M_1(n), M_2(n)$ be the largest, respectively the smallest integer so that $I_{M_1(n)} \subset J_n \subset I_{M_2(n)}$. Then $n = N_2(M_1(n))$, $n = N_1(M_2(n))$, and consequently

$$lim_{n\to\infty} \frac{|J_n| - |I_{M_1(n)}|}{|J_n|} = \lim_{m\to\infty} \frac{|J_{N_2(m)}| - |I_m|}{|J_{N_2(m)}|} = 0$$

since $m = M_1(n) \to \infty$ when $n \to \infty$. Similarly one obtains the other limit

$$\lim_{n\to\infty} \frac{|I_{M_2(n)}| - |J_n|}{|J_n|} = \lim_{m\to\infty} \frac{|I_m| - |J_{N_1(m)}|}{|J_{N_1(m)}|} = 0$$

Now, applying the same argument as before, when $\mathcal{F}_1 \approx \mathcal{F}_2$ with respect to $(I_n)_{n\geq 0}$, one obtains $\mathcal{F}_1 \approx \mathcal{F}_2$ with respect to $(J_n)_{n\geq 0}$. □

2.3. Standard Ordering Revisit

The standard ordering is defined as follows (see[3]). Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$.

DEFINITION 2.10. *We say $\mathcal{F}_1$ is* smaller *than $\mathcal{F}_2$ and write $\mathcal{F}_1 \leq \mathcal{F}_2$ if there is a bounded $T : H_2 \to H_1$ so that $f_i^1 = T f_i^2$ for every $i \in I$.* One can easily check that $\leq$ is indeed a partial ordering relation, that is it is reflexive ($\mathcal{F} \leq \mathcal{F}$), transitive ($\mathcal{F}_1 \leq \mathcal{F}_2$ and $\mathcal{F}_2 \leq \mathcal{F}_3$ imply $\mathcal{F}_1 \leq \mathcal{F}_3$), and antisymmetric ($\mathcal{F}_1 \leq \mathcal{F}_2$ and $\mathcal{F}_2 \leq \mathcal{F}_1$ imply $\mathcal{F}_1 \sim \mathcal{F}_2$). Moreover, it admits the following equivalent geometric characterization:

Theorem 2.11 ([3, 11]). *Consider $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ and P_1, P_2 their associated orthogonal projections onto the coefficients range. Then $\mathcal{F}_1 \leq \mathcal{F}_2$ if and only if $P_1 \leq P_2$ (that is $Ran\, P_1 \subset Ran\, P_2$).* The antisymmetry property with respect to $\sim$ equivalence relation suggests to search and define a new ordering relation that is compatible to $\approx$ equivalence relation. This is accomplished in the following subsection.

2.4. The New Ordering Relation

Fix $(I_n)_{n\geq 0}$ a nested, covering subsequence of finite subsets of I. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ be two frames indexed by I with canonical duals $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$, respectively.

DEFINITION 2.12. *We say $\mathcal{F}_1$ is* more redundant *than $\mathcal{F}_2$, and write $\mathcal{F}_1 \trianglelefteq \mathcal{F}_2$ if*

$$\liminf_{n\to\infty} \frac{1}{|I_n|} \sum_{i\in I_n} (\langle f_i^2, \tilde{f}_i^2\rangle - \langle f_i^1, \tilde{f}_i^1\rangle) \geq 0 \tag{29}$$

It is not hard to check for $\trianglelefteq$ that

$$\mathcal{F} \trianglelefteq \mathcal{F} \quad \text{(reflexive)} \tag{30}$$

$$\mathcal{F}_1 \trianglelefteq \mathcal{F}_2 \ \ and \ \ \mathcal{F}_2 \trianglelefteq \mathcal{F}_3 \Rightarrow \mathcal{F}_1 \trianglelefteq \mathcal{F}_3 \quad \text{(transitive)} \tag{31}$$

$$\mathcal{F}_1 \trianglelefteq \mathcal{F}_2 \ \ and \ \ \mathcal{F}_2 \trianglelefteq \mathcal{F}_1 \Rightarrow \mathcal{F}_1 \approx \mathcal{F}_2 (antisymmetry) \tag{32}$$

Thus $\trianglelefteq$ is a veritable order relation on $\mathcal{F}[I]$.

REMARK 2.13. *Assume $\mathcal{F}_1 \leq \mathcal{F}_2$. Then $P_1 \leq P_2$ in the sense of quadratic forms, that implies $\langle f_i^1, \tilde{f}_i^1\rangle \leq \langle f_i^2, \tilde{f}_i^2\rangle$, hence $\mathcal{F}_1 \trianglelefteq \mathcal{F}_2$. Thus whenever two frames are comparable with respect to $\leq$, they are also comparable with respect to $\trianglelefteq$ and the latter agrees with the former relation.*

An alternative way to introduce a partial ordering that parallels the Murray - von Neumann projection comparison theory is furnished by the following definition. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$.

DEFINITION 2.14. *We write $\mathcal{F}_1 \sqsubseteq \mathcal{F}_2$ if there is $\mathcal{F}_3 \in \mathcal{F}[I]$ so that $\mathcal{F}_1 \approx \mathcal{F}_3$ and $\mathcal{F}_3 \leq \mathcal{F}_2$.* Note the following immediate properties of $\sqsubseteq$:

$$\mathcal{F} \sqsubseteq \mathcal{F} \quad \text{(reflexivity)} \tag{33}$$

$$\mathcal{F}_1 \sqsubseteq \mathcal{F}_2 \ \ and \ \ \mathcal{F}_2 \sqsubseteq \mathcal{F}_1 \Rightarrow \mathcal{F}_1 \approx \mathcal{F}_2 \quad \text{(antisymmetry)} \tag{34}$$

Whether $\sqsubseteq$ has or has not the transitivity property, is not an obvious issue. In fact it still remains an open problem. However the following is true:

Theorem 2.15. *Assume $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$. Then*

1. *If $\mathcal{F}_1 \sqsubseteq \mathcal{F}_2$ then $\mathcal{F}_1 \trianglelefteq \mathcal{F}_2$;*
2. *Assume $\sqsubseteq$ is transitive. Then $\mathcal{F}_1 \trianglelefteq \mathcal{F}_2$ implies $\mathcal{F}_1 \sqsubseteq \mathcal{F}_2$.*

In effect this result shows that, if $\sqsubseteq$ is transitive, hence if it is a partial ordering relation on $\mathcal{F}[I]$, it is equivalent to $\trianglelefteq$. Since out theory is based on $\trianglelefteq$, it will be little affected by a future solution to the transitivity issue of $\sqsubseteq$. We shall return to this issue after we present the proof of Theorem 2.15

Proof of Theorem 2.15

1. Assume $\mathcal{F}_1 \approx \mathcal{F}_3 \leq \mathcal{F}_2$. Then

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{|I_n|} \sum_{i\in I_n} (\langle f_i^2, \tilde{f}_i^2\rangle - \langle f_i^1, \tilde{f}_i^1\rangle) &\geq \liminf_{n\to\infty} \frac{1}{|I_n|} \sum_{i\in I_n} (\langle f_i^2, \tilde{f}_i^2\rangle - \langle f_i^3, \tilde{f}_i^3\rangle) \\ + \liminf_{n\to\infty} \frac{1}{|I_n|} \sum_{i\in I_n} (\langle f_i^3, \tilde{f}_i^3\rangle - \langle f_i^1, \tilde{f}_i^1\rangle) &= \liminf_{n\to\infty} \frac{1}{|I_n|} \sum_{i\in I_n} (\langle f_i^2, \tilde{f}_i^2\rangle - \langle f_i^3, \tilde{f}_i^3\rangle) \geq 0\end{aligned}$$

which proves $\mathcal{F}_1 \trianglelefteq \mathcal{F}_2$.

2. Assume $\sqsubseteq$ is transitive and $\mathcal{F}_1 \trianglelefteq \mathcal{F}_2$. Similar to the proof of Theorem 2.7 construct $\mathcal{G}_1 = \{g_i^1; i \in I\}$ and $\mathcal{G}_2 = \{g_i^2; i \in I\}$ so that $\mathcal{F}_1 \approx \mathcal{G}_1$, $\mathcal{F}_2 \approx \mathcal{G}_2$, $a_n(\mathcal{G}_1) \leq a_n(\mathcal{F}_1)$, $a_n(\mathcal{F}_2) \leq a_n(\mathcal{G}_2)$, $a_n(\mathcal{G}_1) \leq a_n(\mathcal{G}_2)$ and g_i^1, g_i^2 are either 0, or distinct elements from an orthonormal set B so that when $g_i^2 = 0$ then $g_i^1 = 0$, and when $g_i^1 \neq 0$,

$g_i^2 = g_i^1 \in B$. Clearly $\mathcal{G}_1 \leq \mathcal{G}_2$. On the one hand $\mathcal{F}_1 \sqsubseteq \mathcal{G}_2$ because $\mathcal{F}_1 \approx \mathcal{G}_1 \leq \mathcal{G}_2$. On the other hand $\mathcal{G}_2 \sqsubseteq \mathcal{F}_2$, trivially because $\mathcal{G}_2 \approx \mathcal{F}_2 \leq \mathcal{F}_2$. Now since $\sqsubseteq$ is transitive we obtain $\mathcal{F}_1 \sqsubseteq \mathcal{F}_2$ which concludes the proof. □

REMARK 2.16. *Theorem 2.15 shows that the binary relation $\sqsubseteq$ would not give a new classification of the classes of equivalent frames. Instead it would also provide another description of the partial order relation $\trianglelefteq$, description similar to Murray-von Neumann projection theory.*

Let us describe in more details the transitivity question for $\sqsubseteq$. Since the equivalence relation $\approx$ on frames induces an equivalence relation on projection operators of $l^2(I)$, we shall state the problem in $l^2(I)$ space. Assume P_1, P_2 are two orthogonal projections in $l^2(I)$ so that $\langle P_1\delta_i, \delta_i\rangle \leq \langle P_2\delta_i, \delta_i\rangle$, for all $i \in I$. Then the transitivity of $\sqsubseteq$ is equivalent to the construction of an orthogonal projection P_3 in $l^2(I)$ such that it is a subprojector of P_2 (that is $Ran\, P_3 \subset Ran\, P_2$) and satisfies:

$$\lim_{n\to\infty} \frac{1}{|I_n|} \sum_{i\in I_n} (\langle P_3\delta_i, \delta_i\rangle - \langle P_1\delta_i, \delta_i\rangle) = 0 \tag{35}$$

3. THE FRAME MEASURE FUNCTION

In this section we present the definition and basic properties of the frame measure function, followed by the existence and universality of the canonical free ultrafilter frame measure function.

Again I denotes a countable index set, $\mathcal{F}[I]$ denotes the set of frames indexed by I and we fix $(I_n)_{n\geq 0}$ a nested, covering sequence of finite subsets of I.

We recall the notion of superframe (see[1, 2, 4]) (or disjoint frames, as used by D.Larson, see[19]). Let $\mathcal{F}_1, \ldots, \mathcal{F}_L \in \mathcal{F}[I]$, a finite number of frames indexed by I.

DEFINITION 3.1. *We call $(\mathcal{F}_1, \ldots, \mathcal{F}_L)$ a* superframe *if*

$$\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_L := \{f_i^1 \oplus \cdots \oplus f_i^L \ ; \ i \in I\} \tag{36}$$

is a frame in $H_1 \oplus \cdots \oplus H_L$, the direct sum of Hilbert spaces spanned by $\mathcal{F}_1, \ldots, \mathcal{F}_L$, respectively.

An equivalent characterization of superframes is given by the following

Theorem 3.2 ([4]). *The collection $(\mathcal{F}_1, \ldots, \mathcal{F}_L)$ is a superframe if and only if the following two conditions hold true:*

1. *Each $\mathcal{F}_l$ is frame, $1 \leq l \leq L$;*
2. *$E_k \cap (\sum_{l\neq l} E_l) = \{0\}$, for $1 \leq k \leq L$, and $\sum_{k=1}^{L} E_l$ is closed (where E_l is coefficients range in $l^2(I)$ of the analysis operator associated to $\mathcal{F}_l$).*

In particular, the second condition above holds true when the coefficients range E_l are mutually orthogonal. This special case is called *orthogonal in the sense of supersets* (or strongly disjoint, see[19]). More specific we define

DEFINITION 3.3. *Two Bassel sequences $\mathcal{F}_1 = \{f_i^i; i \in I\}$ and $\mathcal{F}_2 = \{f_i^2; i \in I\}$ indexed by I are said* orthogonal in the sense of supersets *if E_1, the range of analysis operator associated to $\mathcal{F}_1$, is orthogonal in $l^2(I)$ to E_2, the range of coefficients associated to $\mathcal{F}_2$.* The condition in the definition reads as:

$$\sum_{i\in I} \langle g, f_i^1\rangle\langle f_i^2, h\rangle = 0 \quad , \quad \forall g \in H_1 \ , \ \forall h \in H_2 \tag{37}$$

REMARK 3.4. *Clearly if two frames $\mathcal{F}_1, \mathcal{F}_2$ are orthogonal in the sense of supersets, then $E_1 \cap E_s = \{0\}$ and $E_1 + E_2$ is closed, hence $(\mathcal{F}_1, \mathcal{F}_2)$ is a superframe. Note in this case the space of coefficients associated to $\mathcal{F}_1 \oplus \mathcal{F}_2$ is exactly $E_1 \oplus E_2$, and the orthogonal projection onto this space, P is given by $P = P_1 + P_2$, the sum of orthogonal* *projections associated to* $\mathcal{F}_1$*, respectively* $\mathcal{F}_2$*. In particular, the canonical dual of* $\mathcal{F}_1 \oplus \mathcal{F}_2$ *is the direct sum of the canonical duals of* $\mathcal{F}_1$ *and* $\mathcal{F}_2$*.*

REMARK 3.5. *For any frame* $\mathcal{F} \in \mathcal{F}[I]$*, one can always construct* $\mathcal{F}' \in \mathcal{F}[I]$ *that is orthogonal to* $\mathcal{F}$ *in the sense of supersets. Indeed, this is done as follows. Let* P *be the projection in* $l^2(I)$ *onto the range of the analysis operator associated to* $\mathcal{F}$*. Then* $Q = I - P$ *is also an orthogonal projection in* $l^2(I)$*. Project now the canonical basis of* $l^2(I)$ *onto* $Ran\,Q$*, say* $\mathcal{F}' = \{Q\delta_i\ ;\ i \in I\}$*. One can easily chack now that* $\mathcal{F}'$ *is a (normalized tight) frame and the range of ita analysis operator is exactly* $Ran\,Q$*, therefore orthogonal to* $\mathcal{F}$ *in the sense of supersets.*

EXAMPLE 3.6. *Consider the following model, namely the set of* Fourier frames. *In* $H = L^2[0,1]$ *consider the Fourier basis* $e_n(x) = e^{2\pi i n x}$*,* $n \in \mathbf{Z}$*. Consider an invertible mapping* $r : I \to \mathbf{Z}$*, and, by abuse of notation, redefine* $e_i = e_{r(i)}$*. For every measurable subset* $J \subset [0,1]$ *denote* $f_i^J = e_i 1_J$ *the product between* e_i *and the characteristic function of* J*,* 1_J*. It is easy to check that* $\mathcal{F}_J = \{f_i^J\ ;\ i \in I\}$ *is a normalized tight frame indexed by* I *whose closed span is* $L^2(J)$*. Moreover, for two subsets* $J_1, J_2 \subset [0,1]$ *so that* $J_1 \cap J_2 = \emptyset$*,* $\mathcal{F}_{J_1}$ *and* $\mathcal{F}_{J_2}$ *are orthogonal frames in the sense of supersets and* $\mathcal{F}_{J_1} \oplus \mathcal{F}_{J_2} \sim \mathcal{F}_{J_1 \cup J_2}$*. On the other hand, for* $J_1 \subset J_2$*, then* $\mathcal{F}_{J_1} \leq \mathcal{F}_{J_2}$ *and therefore* $\mathcal{F}_{J_1} \trianglelefteq \mathcal{F}_{J_2}$*. Note also* $\|f_i^J\|^2 = \mu(J)$*, the Lebesgue measure of* J*. Therefore for two subsets* J_1, J_2 *so that* $\mu(J_1) = \mu(J_2)$*,* $\mathcal{F}_{J_1} \approx \mathcal{F}_{J_2}$*.*

3.1. Definition of the Frame Measure and Redundancy Functions

Fix $(I_n)_{n \geq 0}$ a nested, covering sequence of finite subsets of I.

For a compact space M, we denote by $\mathcal{C}^*(M)$ the set of real-valued continuous functions over M, and $\mathcal{C}(M)$ the set of complex-valued continuous functions over M:

$$\mathcal{C}^*(M) = \{f : M \to \mathbf{R} \mid f \text{ continuous }\} \tag{38}$$

$$\mathcal{C}(M) = \{f : M \to \mathbf{C} \mid f \text{ continuous }\} \tag{39}$$

If M is also Hausdorff separable, then it is normal (see Chapter 1, Section 2.8 in[25]) and by Urysohn's Lemma for any two disjoint closed subsets F_0, F_1 of M, there is a continuous real-valued function $f \in \mathcal{C}^*(M)$ so that i) $0 \leq f(x) \leq 1$, for all $x \in M$; ii) $f|_{F_0} = 0$; iii) $f|_{F_1} = 1$.

A frame measure function is a representation of the set of frames indexed by the same index set with a predefined summation order $(I_n)_{n \geq 0}$, that is faithfull with respect to the equivalence relation $\approx$ (first property), compatible with the partial ordering $\trianglelefteq$ (second property), normalized on s-Riesz bases (third property), and additive on orthogonal superframes (fourth property). More specific,

DEFINITION 3.7. *Given a compact and Hausdorff separable space* M*, we call* $m : \mathcal{F}[I] \to \mathcal{C}^*(M)$ *a* frame measure function *if it satisfies the following properties:*

1. *For any* $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$*,* $\mathcal{F}_1 \approx \mathcal{F}_2$ *if and only if* $m(\mathcal{F}_1) = m(\mathcal{F}_2)$*;*
2. *For any* $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$*,* $\mathcal{F}_1 \trianglelefteq \mathcal{F}_2$ *if and only if* $m(\mathcal{F}_1) \leq m(\mathcal{F}_2)$*;*
3. *For any Riesz basis for its span* $\mathcal{G} \in \mathcal{F}[I]$*,* $m(\mathcal{G}) = 1$*, the constant function* 1 *over* M*;*
4. *For any* $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ *so that* $(\mathcal{F}_1, \mathcal{F}_2)$ *are orthogonal in the sense of supersets,* $m(\mathcal{F}_1 \oplus \mathcal{F}_2) = m(\mathcal{F}_1) + m(\mathcal{F}_2)$*.*

Frame redundancy is measured by the inverse of a frame measure function. More specifically,

DEFINITION 3.8. *A map* $R : \mathcal{F}[I] \to \overline{\mathcal{C}^*(M)}$*, where* $\overline{\mathcal{C}^*(M)} = \{f : M \to \mathbf{R} \cup \{\infty\} \mid f \ continuous\ where\ is\ finite\}$*, is called a* frame redundancy function*, if* $m : \mathcal{F}[I] \to \mathcal{C}^*(M)$*,* $m(\mathcal{F})(p) = 1/R(\mathcal{F})(p)$*, is a frame measure function.*

REMARK 3.9. *One can ask whether properties 3,4 can be strengthen. More specifically, one can ask whethere there is a frame measure function so that if* $m(\mathcal{F}) = 1$*, then* $\mathcal{F}$ *is necessarily a Riesz basis for its span, and, or, if* $(\mathcal{F}_1, \mathcal{F}_2)$ *is a superframe (not necessarily superorthogonal), then* $m(\mathcal{F}_1 \oplus \mathcal{F}_2) = m(\mathcal{F}_1) + m(\mathcal{F}_2)$*. The answer* *to these questions is negative. Before we give the complete answer, we need to prove several properties of frame measure functions.*

In the following we shall present several properties of frame measure functions. We postpone an explicit construction of a frame measure function until the next subsection.

The following properties are stated in a sequence of propositions. We assume $m : \mathcal{F}[I] \to \mathcal{C}^*(M)$ is a frame measure function.

Proposition 3.10. *For any $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_D \in \mathcal{F}[I]$ frames that are mutually orthogonal in the sense of supersets, $m(\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_D) = m(\mathcal{F}_1) + \cdots + m(\mathcal{F}_D)$.*

The proof is immediate by induction.

Proposition 3.11. *For any $\mathcal{F} \in \mathcal{F}[I]$, $0 \leq m(\mathcal{F})(p) \leq 1$ for every $p \in M$.*

Proof

Since $\mathcal{F} \leq \mathcal{G}$ for any orthonormal set $\mathcal{G} \in \mathcal{F}[I]$, from properties 2 and 3 it follows that $m(\mathcal{F})(p) \leq 1$. On the other hand, for any frame $\mathcal{F} \in \mathcal{F}[I]$, one can always construct $\mathcal{F}' \in \mathcal{F}[I]$ that is orthogonal to $\mathcal{F}$ in the sense of supersets (see Remark 3.5). Moreover $\mathcal{F} \oplus \mathcal{F}'$ is a Riesz basis for its span $H \oplus H'$, since its coefficients range is the full space $l^2(I)$. Then $1 = m(\mathcal{F} \oplus \mathcal{F}') = m(\mathcal{F}) + m(\mathcal{F}')$ and since $m(\mathcal{F}') \leq 1$ we conclude that $m(\mathcal{F}) \geq 0$. Q.E.D.

Proposition 3.12. *Assume for $\mathcal{F} \in \mathcal{F}[I]$, $\lim_{n\to\infty} a_n(\mathcal{F})$ exists. Then*

$$m(\mathcal{F})(p) = \lim_{n\to\infty} \frac{1}{|I_n|} \sum_{i \in I_n} \langle f_i, \tilde{f}_i \rangle \quad , \quad \forall p \in M \tag{40}$$

Proof

The proof of this result uses an explicit realization of frames that have constant averages a_n. Consider the Fourier frame model (see Example 3.6).

Consider $p \in \mathbf{N}$ a nonzero integer. Set $J_1 = [0, \frac{1}{p}]$, $J_2 = [\frac{1}{p}, \frac{2}{p}]$, ..., $J_p = [\frac{p-1}{p}, 1]$ the set of p disjoint intervals of equal length covering $[0, 1]$. Then $\mathcal{F}_{J_1}, \ldots, \mathcal{F}_{J_p}$ are normalized tight frames mutually orthogonal in the sense of supersets , and applying Proposition 3.10,

$$m(\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_p) = \sum_{k=1}^{p} m(\mathcal{F}_{J_k})$$

But the left hand side is 1, since it is the measure of a frame equivalent to the orthonormal basis $J_{[0,1]}$. On the other hand $\mathcal{F}_{J_k} \approx \mathcal{F}_{J_l}$ for and $1 \leq k, l \leq p$. Therefore we obtain $m(\mathcal{F}_{J_k})(x) = \frac{1}{p}$ for all $x \in M$.

Next consider $0 < \frac{q}{p} < 1$ a rational number. Set $K_0 = [0, \frac{q}{p}]$, $K_1 = [\frac{q}{p}, \frac{q+1}{p}]$, ..., $K_{p-q} = [\frac{p-1}{p}, 1]$. Then

$$1 = m(\mathcal{F}_{K_0} \oplus \mathcal{F}_{K_1} \oplus \cdots \oplus \mathcal{F}_{K_{p-q}}) = m(\mathcal{F}_{J_0}) + \sum_{k=1}^{p-q} m(\mathcal{F}_{K_k}) = m(\mathcal{F}_{J_0}) + \frac{p-q}{q}$$

Hence $m(\mathcal{F}_{J_0}) = \frac{q}{p} 1_M$.

Finally, let $r \in [0, 1]$. For any $\varepsilon > 0$, there are two rational numbers $q_1^\varepsilon, q_2^\varepsilon \in [0, 1]$ so that $r - \varepsilon < q_1^\varepsilon \leq r \leq q_2^\varepsilon < r + \varepsilon$. But $\mathcal{F}_{[0,q_1^\varepsilon]} \trianglelefteq \mathcal{F}_{[0,r]} \trianglelefteq \mathcal{F}_{[0,q_2^\varepsilon]}$. Therefore $q_1^\varepsilon = m(\mathcal{F}_{[0,q_1^\varepsilon]}) \leq m(\mathcal{F}_{[0,r]}) \leq m(\mathcal{F}_{[0,q_2^\varepsilon]}) = q_2^\varepsilon$. Since ε was arbitrary we conclude $m(\mathcal{F}_{[0,r]})(x) = r$, for all $x \in M$.

We conclude the proof by noting that, if $r = \lim_{n\to\infty} a_n(\mathcal{F})$ then $\mathcal{F} \approx \mathcal{F}_{[0,r]}$. Hence $m(\mathcal{F})(x) = r$ for all $x \in M$. Q.E.D.

Proposition 3.12 allows us to answer the questions raised in Remark 3.9. This is given through two negative statements:

Proposition 3.13. *There is no map $m : \mathcal{F}[I] \to \mathcal{C}^*(M)$ that satisfies:*

1. *For any* $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$, $\mathcal{F}_1 \approx \mathcal{F}_2$ *if and only if* $m(\mathcal{F}_1) = m(\mathcal{F}_2)$;

2. *For any* $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$, $\mathcal{F}_1 \trianglelefteq \mathcal{F}_2$ *if and only if* $m(\mathcal{F}_1) \leq m(\mathcal{F}_2)$;

3. *A frame* $\mathcal{G} \in \mathcal{F}[I]$ *is a Riesz basis for its span if and only if* $m(\mathcal{G}) = 1$, *the constant function* 1 *over* M;

4. *For any* $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ *so that* $(\mathcal{F}_1, \mathcal{F}_2)$ *are orthogonal in the sense of supersets,* $m(\mathcal{F}_1 \oplus \mathcal{F}_2) = m(\mathcal{F}_1) + m(\mathcal{F}_2)$.

Proof

Assume such a mapping m exists. Then it would be a frame measure function and would have the property that, if $m(\mathcal{G}) = 1$, then necessarily $\mathcal{G}$ is a Riesz basic sequence. Construct $\mathcal{G} = \{g_i; i \in I\}$ where $g_i = e_i$ for $i \in I, i \neq i_1, i_2$, $g_{i_1} = g_{i_2} = \frac{1}{\sqrt{2}} e_{i_1}$, and $\{e_i\}$ is an orthonormal set. Clearly $\mathcal{G}$ is frame that has excess 1, because $\{g_i; i \in I, i \neq i_1\}$ is a Riesz basis for the same span as $\mathcal{G}$. Moreover, $\mathcal{G}$ is a Parseval frame, hence its canonical dual coincides with $\mathcal{G}$. For any sequence $(I_n)_{n \geq 0}$ of finite, nested, and covering subsets of I, the averages $a_n(\mathcal{G})$ are either 1, or $\frac{|I_n| - 0.5}{|I_n|}$, or $\frac{|I_n| - 1}{|I_n|}$. Either way, $\lim_{n \to \infty} a_n(\mathcal{G}) = 1$ which, combined with Proposition 3.12, implies $m(\mathcal{G}) = 1$. Yet $\mathcal{G}$ is not Riesz basic sequence. Q.E.D.

Proposition 3.14. *There is no map* $m : \mathcal{F}[I] \to \mathcal{C}^*(M)$ *that satisfies:*

1. *For any* $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$, $\mathcal{F}_1 \approx \mathcal{F}_2$ *if and only if* $m(\mathcal{F}_1) = m(\mathcal{F}_2)$;

2. *For any* $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$, $\mathcal{F}_1 \trianglelefteq \mathcal{F}_2$ *if and only if* $m(\mathcal{F}_1) \leq m(\mathcal{F}_2)$;

3. *If* $\mathcal{G} \in \mathcal{F}[I]$ *is a Riesz basis for its span then* $m(\mathcal{G}) = 1$, *the constant function* 1 *over* M;

4. *For any* $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ *so that* $(\mathcal{F}_1, \mathcal{F}_2)$ *is a superframe,* $m(\mathcal{F}_1 \oplus \mathcal{F}_2) = m(\mathcal{F}_1) + m(\mathcal{F}_2)$.

Proof Assume such a mapping m exists. Then it would be a frame measure function and would have the property that, if $(\mathcal{F}_1, \mathcal{F}_2)$ is a superframe not necessarily orthogonal in the sense of supersets, then $m(\mathcal{F}_1 \oplus \mathcal{F}_2) = m(\mathcal{F}_1) + m(\mathcal{F}_2)$. We shall construct two frames $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ so that $(\mathcal{F}_1, \mathcal{F}_2)$ is a superframe, yet $m(\mathcal{F}_1 \oplus \mathcal{F}_2) \neq m(\mathcal{F}_1) + m(\mathcal{F}_2)$. To simplify the analysis, consider the case $I = \mathbf{Z} \setminus \{0\}$ and $I_n = [-n, n] \cap I$. Let $\{e_k\}_{k \in \mathbf{Z}}$ be an orthonormal set. Then define $\mathcal{G}_1 = \{g_k^1; k \in I\}$ by $g_k^1 = \frac{1}{2}(e_k + e_{-k})$. Thus $\mathcal{G}_1$ is a Parseval frame for $\overline{span}\{e_1 + e_{-1}, e_2 + e_2, \cdots, e_k + e_{-k}, \cdots\}$ and $\|g_k^1\|^2 = \frac{1}{2}$. Define $\mathcal{G}_2 = \{g_k^2; k \in I\}$ by $g_k^2 = e_k$, $g_{-k}^2 = 0$, for all $k > 0$. Clearly $\mathcal{G}_2$ is also a Parseval frame. A direct verification shows that $(\mathcal{G}_1, \mathcal{G}_2)$ is a supeframe. In fact $\mathcal{G}_1 \oplus \mathcal{G}_2$ is a Riesz basis for $\overline{span}\{e_k; k \in \mathbf{Z} \setminus \{0\}\}$. Now consider an invertible map $\sigma : I \to I$ such that $\lim_{n \to \infty} \frac{1}{2n} |\sigma([-n, n]) \cap \mathbf{N}| = 1$. For instance $\sigma(n) = 2n$, $\sigma(-n) = -2n - 1$, for $n > 0$ that is not a power of 2; for $n = 2^m$, $\sigma_{-n} = -m$, $\sigma_n = c_m$, where $\{c_m\} = \{3, 4, 7, 8, 15, 16, \ldots, 2^k - 1, 2^k, \ldots\}$. Define $\mathcal{F}_1 = \{f_n^1; f_n^1 = g_{\sigma(n)}^1\}$, $\mathcal{F}_2 = \{f_n^2; f_n^2 = g_{\sigma(n)}^2\}$. Note each $\mathcal{F}_1, \mathcal{F}_2$ is a Parseval frame. Moreover, $a_n(\mathcal{F}_1) = \frac{1}{2}$, and $a_n(\mathcal{F}_2) = \frac{1}{2n} |\sigma([-n, n]) \cap \mathbf{N}|$. Therefore $m(\mathcal{F}_1) = \frac{1}{2}$ and $m(\mathcal{F}_2) = 1$. Since the angle between $Ran\, T_{\mathcal{F}_1}$ and $Ran\, T_{\mathcal{F}_2}$ is the same as the angle between $Ran\, T_{\mathcal{G}_1}$ and $Ran\, T_{\mathcal{G}_2}$ since σ induces a unitary (permutation) transformation in $l^2(I)$ that maps $Ran\, T_{\mathcal{G}_1}$ into $Ran\, T_{\mathcal{F}_1}$ and $Ran\, T_{\mathcal{G}_2}$ into $Ran\, T_{\mathcal{F}_2}$. Consequently $(\mathcal{F}_1, \mathcal{F}_2)$ is also a superframe and a Riesz basis. Hence $m(\mathcal{F}_1 \oplus \mathcal{F}_2) = 1$. Clearly $m(\mathcal{F}_1 \oplus \mathcal{F}_2) \neq m(\mathcal{F}_1) + m(\mathcal{F}_2)$. Q.E.D.

REMARK 3.15. *In a private communication,*[9] *Pete Casazza constructed an example so that for every* $\varepsilon > 0$ *there is a superframe* $(\mathcal{F}_1, \mathcal{F}_2)$ *so that each* $\mathcal{F}_1 = \{f_i^1; i \in I\}$ *and* $\mathcal{F}_2 = \{f_i^2; i \in I\}$ *is a Parseval frame,* $norm f_i^1 = c_1$, $\|f_2^i\| = c_2$, *for all* $i \in I$, *and* $1 - \varepsilon < c_1, c_2 < 1$.

REMARK 3.16. *In general* $m(\mathcal{F}_1 \oplus \mathcal{F}_2) \neq m(\mathcal{F}_1) + m(\mathcal{F}_2)$ *when* $(\mathcal{F}_1, \mathcal{F}_2)$ *is a superframe. However the equality holds true only for special cases. By definition, it holds true whenever* $(\mathcal{F}_1, \mathcal{F}_2)$ *are orthogonal in the sense of supersets.*

Proposition 3.17. *For any* $\mathcal{F} \in \mathcal{F}[I]$ *and for every* $x \in M$,

$$\liminf_{n \to \infty} a_n(\mathcal{F}) \leq m(\mathcal{F})(x) \leq \limsup_{n \to \infty} a_n(\mathcal{F}) \tag{41}$$

Proof

Let $r = \liminf_{n\to\infty} a_n(\mathcal{F})$. Consider the Fourier frame $\mathcal{F}_{[0,r]}$. Clearly $\liminf_{n\to\infty}(a_n(\mathcal{F}) - a_n(\mathcal{F}_{[0,r]})) = \liminf_{n\to\infty} a_n(\mathcal{F}) - r \geq 0$. Hence $\mathcal{F}_{[0,r]} \trianglelefteq \mathcal{F}$ and $r \leq m(\mathcal{F})(x)$ for every $x \in M$.

Similarly, if $R = \limsup_{n\to\infty} a_n(\mathcal{F})$ then $\mathcal{F} \trianglelefteq \mathcal{F}_{[0,R]}$ and thus $m(\mathcal{F})(x) \leq R$, for every $x \in M$. Q.E.D.

Proposition 3.18. *For any $\mathcal{F} \in \mathcal{F}[I]$ there are $x_0, y_0 \in M$ so that*

$$m(\mathcal{F})(x_0) = \liminf_{n\to\infty} a_n(\mathcal{F}) \tag{42}$$

$$m(\mathcal{F})(y_0) = \limsup_{n\to\infty} a_n(\mathcal{F}) \tag{43}$$

Proof

We use again the Fourier frames $\mathcal{F}_J$. Let $r = min_{x\in M} m(\mathcal{F})(x)$. Then $r \leq m(\mathcal{F})$ which implies $\mathcal{F}_{[0,r]} \trianglelefteq \mathcal{F}$ (by Axiom 2). Therefore $\liminf_{n\to\infty}(a_n(\mathcal{F}) - r) \geq 0$. Hence $\liminf_{n\to\infty} a_n(\mathcal{F}) \geq r$. Using now Proposition 3.17 we obtain $r = \liminf_{n\to\infty} a_n(\mathcal{F})$. Since $m(\mathcal{F})$ is continuous on the compact M, it follows it achieves its minimum at some point x_0. Hence (42).

A similar argument proves (43). Q.E.D.

Proposition 3.19. *For any $\mathcal{F} \in \mathcal{F}[I]$ and $r \in [0,1]$, there is $\mathcal{F}_r \in \mathcal{F}[I]$ so that $a_n(\mathcal{F}_r) = ra_n(\mathcal{F})$ and $m(\mathcal{F}_r) = rm(\mathcal{F})$.*

Proof

The proof goes along the lines of proof of Proposition 3.12. First use Theorem 2.7 and replace $\mathcal{F}$ equivalently by a frame $\mathcal{G}$ made of an orthonormal set interlaced with zero vectors. For an integer p the frame $\mathcal{G}$ can be decompose into p frames $\mathcal{G}_1, \ldots, \mathcal{G}_p \in \mathcal{F}[I]$ defined by projecting $\mathcal{G}$ onto $H_1, \ldots, H_p$ an orthogonal decomposition of *span*$(\mathcal{G})$ where each H_k are spanned by an average of $\frac{1}{p}$ nonzero vectors of $\mathcal{G}$ (that is for each I_n, about $\frac{1}{p}$ nonzero vectors from $\{g_i; i \in I_n\}$ go to each H_k). Then $\mathcal{G} \sim \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_p$, where $(\mathcal{G}_1, \cdots, \mathcal{G}_p)$ are mutually disjoint, and $\mathcal{G}_1 \approx \cdots \approx \mathcal{G}_p$. Therefore $m(\mathcal{G}) = pm(\mathcal{G}_1)$, hence $m(\mathcal{G}_1) = \frac{1}{p}m(\mathcal{G}) = \frac{1}{p}m(\mathcal{F})$.

Next, for $r = \frac{q}{p}$, we have $m(\mathcal{G}_1 \oplus \cdots \mathcal{G}_q) = qm(\mathcal{G}_1) = \frac{q}{p}m(\mathcal{F})$.

Finally, for $r \notin \mathbf{Q}$, construct $\mathcal{F}_r \leq \mathcal{G}$ so that $|a_n(\mathcal{F}_r) - ra_n(\mathcal{G})| \leq \frac{1}{|I_n|}$ similar to the construction in the proof of Theorem 2.7. Next, for any rational number $q_1 < r$ we can construct $\mathcal{G}_{q_1}$ so that $a_n(\mathcal{G}_{q_1}) \leq a_n(\mathcal{F}_r)$ and $|a_n(\mathcal{G}_{q_1}) - q_1 a_n(\mathcal{F})| \leq \frac{1}{|I_n|}$. Therefore $q_1 m(\mathcal{F}) = m(\mathcal{G}_{q_1}) \leq m(\mathcal{F}_r)$. Similar, for any rational $q_2 > r$ construct $\mathcal{G}_{q_2}$ so that $a_n(\mathcal{G}_{q_2}) \geq a_n(\mathcal{F}_r)$ and $|a_n(\mathcal{G}_{q_2}) - q_2 a_n(\mathcal{F})| \leq \frac{1}{|I_n|}$. Therefore $q_2 m(\mathcal{F}) = m(\mathcal{G}_{q_2}) \geq m(\mathcal{F}_r)$. Since q_1, q_2 were arbitrary, we obtain $m(\mathcal{F}_r) = rm(\mathcal{F})$. Q.E.D.

Proposition 3.20. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_D \in \mathcal{F}[I]$ and $r_1, \cdots, r_D \in [0,1]$ be such that $r_1 + \cdots + r_D = 1$. Then there is $\mathcal{F} \in \mathcal{F}[I]$ so that $m(\mathcal{F}) = r_1 m(\mathcal{F}_1) + \cdots r_D m(\mathcal{F}_D)$ for any frame measure function m.*

Proof

First using Proposition 3.19 we construct $\mathcal{G}_1, \ldots, \mathcal{G}_D \in \mathcal{F}[I]$ so that $m(\mathcal{G}_k) = c_k m(\mathcal{F}_k)$, $1 \leq k \leq D$. Moreover the frames $\mathcal{G}_k$ are in the canonical form given by Theorem 2.7. Since $r_1 a_n(\mathcal{F}_1) + \cdots + r_D a_n(\mathcal{F}_D) \leq 1$ for all n, we can adapt the construction of Theorem 2.7 so that the vectors of $\mathcal{G}_k$ are mutually nonzero, that is if $g_i^{k_0} \neq 0$ for some $1 \leq k_0 \leq D$ and $i \in I$, then $g_i^k = 0$ for all $1 \leq k \leq D$ with $k \neq k_0$. That means $(\mathcal{G}_1, \ldots, \mathcal{G}_D)$ is a superframe of mutually orthogonal frames (in the sense of supersets). Define $\mathcal{F} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_D$. Hence

$$m(\mathcal{F}) = m(\mathcal{G}_1) + \cdots + m(\mathcal{G}_D) = r_1 m(\mathcal{F}_1) + \cdots + r_D m(\mathcal{F}_D).$$

Note $\mathcal{F}$ is independent of the particular measure function m. Q.E.D.

In particular, Proposition 3.20 says that the range of any frame measure function is a convex set.

With respect to $\approx$ equivalent classes of frames have a natural lattice structure that we describe in the following.

Theorem 3.21. *Consider $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$. There are $\mathcal{F}_m, \mathcal{F}_M \in \mathcal{F}[I]$ so that:*

$$\mathcal{F}_m \quad \trianglelefteq \quad \mathcal{F}_M \tag{44}$$

$$min(a_n(\mathcal{F}_1), a_n(\mathcal{F}_2)) - \frac{1}{|I_n|} \leq \quad a_n(\mathcal{F}_m) \quad \leq min(a_n(\mathcal{F}_1), a_n(\mathcal{F}_2)) \tag{45}$$

$$max(a_n(\mathcal{F}_1), a_n(\mathcal{F}_2)) \leq \quad a_n(\mathcal{F}_M) \quad \leq max(a_n(\mathcal{F}_1), a_n(\mathcal{F}_2)) + \frac{1}{|I_n|} \tag{46}$$

Moreover the measures of $\mathcal{F}_m$ and $\mathcal{F}_M$ satisfy:

$$m(\mathcal{F}_m) \quad \leq \quad min(m(\mathcal{F}_1), m(\mathcal{F}_2)) \tag{47}$$

$$m(\mathcal{F}_M) \quad \geq \quad max(m(\mathcal{F}_1), m(\mathcal{F}_2)) \tag{48}$$

$$m(\mathcal{F}_m) + m(\mathcal{F}_M) \quad = \quad m(\mathcal{F}_1) + m(\mathcal{F}_2) \tag{49}$$

for any measure function m. This result suggests the following definition:

Definition 3.22. *We denote by $\mathcal{F}_1 \wedge \mathcal{F}_2$ the frame class of $\mathcal{F}_m$, and by $\mathcal{F}_1 \vee \mathcal{F}_2$ the frame class $\mathcal{F}_M$ above.* One can easily check for three frames $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \mathcal{F}[I]$ that:

$$\mathcal{F}_1 \wedge (\mathcal{F}_2 \wedge \mathcal{F}_3) \quad \approx \quad (\mathcal{F}_1 \wedge \mathcal{F}_2) \wedge \mathcal{F}_3 \tag{50}$$

$$\mathcal{F}_1 \vee (\mathcal{F}_2 \vee \mathcal{F}_3) \quad \approx \quad (\mathcal{F}_1 \vee \mathcal{F}_2) \vee \mathcal{F}_3 \tag{51}$$

$$\mathcal{F}_1 \wedge (\mathcal{F}_2 \vee \mathcal{F}_3) \quad \approx \quad (\mathcal{F}_1 \wedge \mathcal{F}_2) \vee (\mathcal{F}_1 \wedge \mathcal{F}_3) \tag{52}$$

$$\mathcal{F}_1 \vee (\mathcal{F}_2 \wedge \mathcal{F}_3) \quad \approx \quad (\mathcal{F}_1 \vee \mathcal{F}_2) \wedge (\mathcal{F}_1 \vee \mathcal{F}_3) \tag{53}$$

Remark 3.23. *Relations (50-53) make $(\mathcal{F}[I]/\approx, \trianglelefteq, \wedge, \vee)$ a distributive lattice of classes of frames (see,[22,26]).*

Proof of Theorem 3.21

Frames $\mathcal{F}_m, \mathcal{F}_M$ that satisfy (44,45, 46) are easily constructed similar to the frames $\mathcal{G}_1, \mathcal{G}_2$ of Theorem 2.7. Indeed, with the notations of the proof of Theorem 2.7, we construct $\mathcal{F}_m$ similar to $\mathcal{G}_1$ so that

$$a_n(\mathcal{F}_m) = \lfloor min(\sum_{i \in I_n} \langle f_i^1, \tilde{f}_i^1 \rangle, \sum_{i \in I_n} \langle f_i^2, \tilde{f}_i^2 \rangle) \rfloor,$$

and $\mathcal{F}_M$ similar to $\mathcal{G}_2$ so that

$$a_n(\mathcal{F}_M) = \lceil max(\sum_{i \in I_n} \langle f_i^1, \tilde{f}_i^1 \rangle, \sum_{i \in I_n} \langle f_i^2, \tilde{f}_i^2 \rangle) \rceil.$$

This proves (45) and (46). Note by construction that $\mathcal{F}_m \trianglelefteq \mathcal{F}_M$, hence (44).

Since $a_n(\mathcal{F}_m) \leq a_n(\mathcal{F}_1)$ and $a_n(\mathcal{F}_m) \leq a_n(\mathcal{F}_2)$, it follows $m(\mathcal{F}_m) \leq m(\mathcal{F}_1)$, $m(\mathcal{F}_m) \leq m(\mathcal{F}_2)$ that is (47). Similarly $a_n(\mathcal{F}_M) \geq a_n(\mathcal{F}_1)$ and $a_n(\mathcal{F}_M) \geq a_n(\mathcal{F}_2)$ imply $m(\mathcal{F}_M) \geq m(\mathcal{F}_1)$ and $m(\mathcal{F}_M) \geq m(\mathcal{F}_2)$, which is (48). Assume now that $a_n(\mathcal{F}_1), a_n(\mathcal{F}_2) \leq \frac{1}{2}$. Then $\mathcal{F}_m$ and $\mathcal{F}_M$ can be chosen so that the set of nonzero vectors are disjoint, that is $(\mathcal{F}_m, \mathcal{F}_M)$ are orthogonal in the sense of supersets. Moreover, $\mathcal{F}_1$ and $\mathcal{F}_2$ can be replaced equivalently by two frames $\mathcal{G}_1$ and $\mathcal{G}_2$ as in Theorem 2.7 that are orthogonal to one another in the sense of supersets. Then $a_n(\mathcal{F}_m) \oplus \mathcal{F}_M) = a_n(\mathcal{F}_m) + a_n(\mathcal{F}_M)$, $a_n(\mathcal{G}_1 \oplus \mathcal{G}_2) = a_n(\mathcal{F}_1) + a_n(\mathcal{F}_2)$ and $\lim_{n \to \infty} |a_n(\mathcal{F}_m \oplus \mathcal{F}_M) - a_n(\mathcal{G}_1 \oplus \mathcal{G}_2)| = 0$. Hence $m(\mathcal{F}_m \oplus \mathcal{F}_M) = m(\mathcal{G}_1 \oplus \mathcal{G}_2)$ and then $m(\mathcal{F}_m) + m(\mathcal{F}_M) = m(\mathcal{F}_1) + m(\mathcal{F}_2)$ which proves (49). The general case when $a_n(\mathcal{F}_1)$ or $a_n(\mathcal{F}_2)$ is greater than $\frac{1}{2}$ reduces to the previous case using Proposition 3.19. Q.E.D.

The following result says that any frame measure function is continuous with respect to the frame topolgy τ introduced at the end of section 2.1. This is a striking result since à priori we have not assumed any continuity property of frame measure functions. It turns out the algebraic conditions are powerful enough to guarantee, in fact, Lipschitz continuity.

Theorem 3.24. *Any frame measure function $m : \mathcal{F}[I] \to \mathcal{C}^*(M)$ is uniformly continuous with respect to τ topology on $\mathcal{F}[I]$, and sup-topology on $\mathcal{C}^*(M)$. Furthermore,for any two frames $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}[I]$ with P_1, P_2 their associated orthonormal projections,*

$$|m(\mathcal{F}_1)(x) - m(\mathcal{F}_2)(x)| \leq 2\|P_1 - P_2\| \tag{54}$$

Proof

First, if $\mathcal{F}_1 \approx \mathcal{F}_2$, then $m(\mathcal{F}_1) = m(\mathcal{F}_2)$ and (54) is satisfied. Therefore consider the case $\mathcal{F}_1 \not\approx \mathcal{F}_2$. Consider $\mathcal{F}_M = \mathcal{F}_1 \vee \mathcal{F}_2$ and $\mathcal{F}_m = \mathcal{F}_1 \wedge \mathcal{F}_2$. Let $b_n = a_n(\mathcal{F}_m)$ and $c_n = a_n(\mathcal{F}_M)$. By construction, $b_n \leq c_n$ and $\limsup(c_n - b_n) > 0$ because we excluded $\mathcal{F}_1 \approx \mathcal{F}_2$. Denote $s_n = b_n|I_n| - b_{n-1}|I_{n-1}|$, $r_n = c_n|I_n| - c_{n-1}|I_{n-1}|$. Note that s_n, r_n are nonnegative integers and

$$|s_n - r_n| \leq 2 + \sum_{i \in I_n \setminus I_{n-1}} |\langle f_i^1, \tilde{f}_i^1\rangle - \langle f_i^2, \tilde{f}_i^2\rangle| \leq 2 + (|I_n| - |I_{n-1}|)\|P_1 - P_2\| \tag{55}$$

$$|\sum_{k=n_1+1}^{n_2} (s_k - r_k)| \leq 2 + \sum_{i \in I_{n_2} \setminus I_{n_1}} |\langle f_i^1, \tilde{f}_i^1\rangle - \langle f_i^2, \tilde{f}_i^2\rangle| \leq 2 + (|I_{n_2}| - |I_{n_1}|)\|P_1 - P_2\| \tag{56}$$

We construct now two new frames $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{F}[I]$, $\mathcal{G}_1 \leq \mathcal{G}_2$, of vectors from an orthonormal set and zeros as follows. Set $\mathcal{G}_1 = \dot\cup_{n\geq 1}\mathcal{G}_n^1$, $\mathcal{G}_2 = \dot\cup_{n\geq 1}\mathcal{G}_n^2$ disjoint decompositions of $\mathcal{G}_1, \mathcal{G}_2$ induced by $(I_n)_{n\geq 1}$, that is $\mathcal{G}_n^1 = \{g_i^1 \ ; \ i \in I_n \setminus I_{n-1}\}$, $\mathcal{G}_i^2 = \{g_i^2 \ ; \ i \in I_n \setminus I_{n-1}\}$. Next we shall construct $\mathcal{G}_n^1, \mathcal{G}_n^2$. For $n = 1$, construct $\mathcal{G}_1^1, \mathcal{G}_1^2$ so that $\mathcal{G}_1^1 \subset \mathcal{G}_1^2$ and $a_1(\mathcal{G}_1^1) = a_1(\mathcal{F}_m)$, $a_1(\mathcal{G}_1^2) = a_1(\mathcal{F}_M)$. For $n > 1$ we construct inductively. Set $d_n = a_n(\mathcal{G}_2) - a_n(\mathcal{G}_1) - (c_n - b_n)$. Note $d_1 = 0$. Assume $\mathcal{G}_k^1, \mathcal{G}_k^2$, $1 \leq k \leq n$, are constructed. Furthermore, assume $a_n(\mathcal{G}^1) \leq b_n$, $a_n(\mathcal{G}^2) \geq c_n$, and thus $d_n \geq 0$. We construct $\mathcal{G}_{n+1}^1$, $\mathcal{G}_{n+1}^2$ as follows. If

$$r_{n+1} - |I_n|(a_n(\mathcal{G}^2) - c_n) \geq |I_n|(b_n - a_n(\mathcal{G}_1)) + s_{n+1} \tag{57}$$

then choose $\mathcal{G}_{n+1}^1$, $\mathcal{G}_{n+1}^2$ so that the cardinal of nonzero vectors of these sets, $\#|\mathcal{G}_{n+1}^1|$, $\#|\mathcal{G}_{n+1}^2|$ are given by

$$\#|\mathcal{G}_{n+1}^1| = |I_n|(b_n - a_n(\mathcal{G}_1)) + s_{n+1}$$
$$\#|\mathcal{G}_{n+1}^2| = r_{n+1} - |I_n|(a_n(\mathcal{G}^2) - c_n)$$

Otherwise, choose $\mathcal{G}_{n+1}^1$, $\mathcal{G}_{n+1}^2$ so that the cardinal of nonzero vectors of these sets, $\#|\mathcal{G}_{n+1}^1|$, $\#|\mathcal{G}_{n+1}^2|$ are given by

$$\#|\mathcal{G}_{n+1}^1| = \#|\mathcal{G}_{n+1}^2| = max(r_{n+1} - |I_n|(a_n(\mathcal{G}^2) - c_n), 0)$$

First we note these sets are realizable because $0 \leq \#|\mathcal{G}_{n+1}^1| \leq \#|\mathcal{G}_{n+1}^2 \leq r_{n+1} \leq |I_{n+1}| - |I_n|$. Next note

$$a_{n+1}(\mathcal{G}_1) = \frac{|I_n|}{|I_{n+1}|}a_n(\mathcal{G}_1) + \frac{\#|\mathcal{G}_{n+1}^1|}{|I_{n+1}|} \leq \frac{|I_n|}{|I_{n+1}|}b_n + \frac{s_{n+1}}{|I_{n+1}|} = b_{n+1}$$

$$a_{n+1}(\mathcal{G}_2) = \frac{|I_n|}{|I_{n+1}|}a_n(\mathcal{G}_2) + \frac{\#|\mathcal{G}_{n+1}^2|}{|I_{n+1}|} \geq \frac{|I_n|}{|I_{n+1}|}c_n + \frac{r_{n+1}}{|I_{n+1}|} = c_{n+1}$$

We thus obtain frames $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{F}[I]$ so that $\mathcal{G}_1 \leq \mathcal{G}_2$, $a_n(\mathcal{G}_1) \leq b_n$, $a_n(\mathcal{G}_2) \geq c_n$, and thus $d_{n+1} \geq 0$. Note the condition (57) can be compactly restated as: $r_{n+1} - s_{n+1} \geq |I_n|d_n$. For d_n we get the following recurrence:

$$d_{n+1} = \begin{cases} 0 & , \text{ if } r_{n+1} - s_{n+1} \geq |I_n|d_n \\ \frac{|I_n|}{|I_{n+1}|}d_n - \frac{r_{n+1}-s_{n+1}}{|I_{n+1}|} & , \ \textit{otherwise} \end{cases}$$

Let $(m_k)_{k\geq 0}$ be the sequence of zeros of d_n, that is $d_{m_k} = 0$. We prove now that $(m_k)_{k\geq 0}$ is not finite. Indeed, assume there is M so that $d_n > 0$ for all $n > M$. Then:

$$d_n = \frac{1}{|I_n}|d_M - \frac{1}{|I_n|}\sum_{k=M+1}^{n}(r_k - s_k) = \frac{1}{|I_n|}(d_M + (c_M - b_M)|I_M|) - (c_n - b_n)$$

Therefore:

$$0 \le \liminf_n d_n = -\limsup_n (c_n - d_n) < 0$$

which is a contradiction. Hence (m_k) is an infinite sequence. Consider now $m_j \le n < m_{j+1}$. Then using the recurrence relation above and (56)

$$d_n = \frac{1}{|I_n|}\sum_{k=m_j+1}^{n}(s_k - r_k) \le \frac{1}{|I_n|}|\sum_{k=m_j+1}^{n}(s_k - r_k)| \quad (58)$$

$$\le \frac{2}{|I_n|} + \frac{|I_n| - |I_{m_j}|}{|I_n|}\|P_1 - P_2\| \le \frac{2}{|I_n|} + \|P_1 - P_2\| \quad (59)$$

Now we return to $\mathcal{F}_1, \mathcal{F}_2$. Note first $|a_n(\mathcal{F}_1) - a_n(\mathcal{F})| \le \|P_1 - P_2\|$. Since $\mathcal{G}_1 \trianglelefteq \mathcal{F}_1$, $\mathcal{G}_1 \trianglelefteq \mathcal{F}_2$ we have $m(\mathcal{G}_1) \le min(m(\mathcal{F}_1), m(\mathcal{F}_2))$. Similarly, from $\mathcal{F}_1 \trianglelefteq \mathcal{G}_2$ and $\mathcal{F}_2 \trianglelefteq \mathcal{G}_2$ we have $m(\mathcal{G}_2) \ge max(m(\mathcal{F}_1), m(\mathcal{F}_2))$. Furthermore, since $\mathcal{G}_1 \le \mathcal{G}_2$, there is $\mathcal{G} \in \mathcal{F}[I]$ so that $\mathcal{G}_2 \sim \mathcal{G}_1 \oplus \mathcal{G}$ and $(\mathcal{G}_1, \mathcal{G})$ is an orthogonal superframe. Then:

$$|m(\mathcal{F}_1)(x) - m(\mathcal{F}_2)(x)| \le |m(\mathcal{G}_2)(x) - m(\mathcal{G}_1)(x)| = m(\mathcal{G})(x) \le \limsup_{n\to\infty} a_n(\mathcal{G})$$

$$= \limsup_{n\to\infty}(a_n(\mathcal{G}_2) - a_n(\mathcal{G}_1)) \le \limsup_{n\to\infty} d_n + \limsup_{n\to\infty}|a_n(\mathcal{F}_1) - a_n(\mathcal{F}_2)| \le 2\|P_1 - P_2\|$$

which proves the Theorem. Q.E.D.

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