Sparse Factorizations of Symmetric Matrices and Decompositions of Trace-Class Operators

Radu Balan

Department of Mathematics, CSCAMM and NWC
University of Maryland, College Park, MD

June 27, 2019
Aspects of Time-Frequency Analysis (ATFA19)
Politecnico di Torino, DISMA, June 25–27 2019, Torino, ITALY

”This material is based upon work supported by the National Science Foundation under Grant No. DMS-1816608. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.” The author has been partially supported by LTS under grant H9823013D00560049.

Collaborators: Kasso Okoudjou (UMD), Anirudha Poria (IIT Guwahati), Michael Rawson (UMD), Yang Wang (HKUST).

Table of Contents:

Problem Formulation
Function Space Formulation

Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a linear operator of the form:

$$Tf(x) = \int_{-\infty}^{\infty} K(x,y)f(y)dy.$$

Assume the following hold true:

1. Kernel $K \in M^1(\mathbb{R}^2)$ belongs to the *modulation space* M^1 (a.k.a. the Feichtinger algebra, or the Segal algebra for the algebra of TF ops). Note: This assumption imples that T is a trace-class compact operator.
2. T is self-adjoint, i.e., $K(x,y) = \overline{K(y,x)}$, for every $x, y, \in \mathbb{R}$;
3. T is positive semi-definite, i.e., $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x,y)f(y)\overline{f(x)}dydx \geq 0$, for every $f \in L^2(\mathbb{R})$. Note: Assumption 2 is redundant in the complex case.

In this talk we study rank-1 series expansions of $T = \sum_k g_k g_k^* := \sum_k \langle \cdot, g_k \rangle g_k$ that satisfy certain convergence properties.

Problem Formulation
Function Space Formulation

The starting point of this study is a problem stated by H. Feichtinger at a 2004 Oberwolfach mini-workshop., and then reformulated and extended by Heil and Larson (2004, 2008).

Let $(f_k)_{k\geq 0}$ be an orthogonal set of eigenfunctions, normalized so that $Tf_k = \|f_k\|_2^2 f_k$ and $T = \sum_k f_k f_k^*$. Then

$$tr(T) = \sum_{k\geq 0} \|f_k\|_2^2 = \sum_{k\geq 0} \|f_k\|_{M^2}^2 \leq \|K\|_{M^1} < \infty.$$

Fact: It is known [HeilLars04/08] that $f_k \in M^1(\mathbb{R})$ for each k.

Problem Formulation

Function Space Formulation

The starting point of this study is a problem stated by H. Feichtinger at a 2004 Oberwolfach mini-workshop., and then reformulated and extended by Heil and Larson (2004, 2008).

Let $(f_k)_{k\geq 0}$ be an orthogonal set of eigenfunctions, normalized so that $Tf_k = \|f_k\|_2^2 f_k$ and $T = \sum_k f_k f_k^*$. Then

$$tr(T) = \sum_{k\geq 0} \|f_k\|_2^2 = \sum_{k\geq 0} \|f_k\|_{M^2}^2 \leq \|K\|_{M^1} < \infty.$$

Fact: It is known [HeilLars04/08] that $f_k \in M^1(\mathbb{R})$ for each k.

Problem 1 [Feichtinger2004]: Does $\sum_{k\geq 0} \|f_k\|_{M^1}^2 < \infty$?

Problem Formulation

Function Space Formulation

The starting point of this study is a problem stated by H. Feichtinger at a 2004 Oberwolfach mini-workshop., and then reformulated and extended by Heil and Larson (2004, 2008).

Let $(f_k)_{k\geq 0}$ be an orthogonal set of eigenfunctions, normalized so that $Tf_k = \|f_k\|_2^2 f_k$ and $T = \sum_k f_k f_k^*$. Then

$$tr(T) = \sum_{k\geq 0} \|f_k\|_2^2 = \sum_{k\geq 0} \|f_k\|_{M^2}^2 \leq \|K\|_{M^1} < \infty.$$

Fact: It is known [HeilLars04/08] that $f_k \in M^1(\mathbb{R})$ for each k.

Problem 1 [Feichtinger2004]: Does $\sum_{k\geq 0} \|f_k\|_{M^1}^2 < \infty$?

Problem 2 [HeilLarson04]: If the answer is negative to Problem 1, is there a decomposition $T = \sum_k g_k g_k^*$, not necessarily spectral, so that $\sum_{k\geq 0} \|g_k\|_{M^1}^2 < \infty$?

Overview of results

I. We construct explicitely an operator T with simple functions that satisfies the previous assumptions and additionally:

1. Its eigenfunctions $(f_k)_{k\geq 0}$ satisfy $\sum_{k\geq 0}\|f_k\|_{M^1}^2=\infty$.
2. There exists a decomposition $T=\sum_{k\geq 0} g_k g_k^*$ so that $\sum_{k\geq 0}\|g_k\|_{M^1}^2<\infty$

II. We introduce a finite-dimensional inequality/hypothesis. We prove the following results:

1. If the hypothesis is false then there exists a non-negative operator T with kernel in M^1 that does not admit a decomposition $T=\sum_{k\geq 0} g_k g_k^*$ so that $\sum_{k\geq 0}\|g_k\|_{M^1}^2<\infty$.
2. On the other hand, if the hypothesis is true, then the set of non-negative operators T with kernel in M^1 that admit a decomposition $T=\sum_{k\geq 0} g_k g_k^*$ so that $\sum_{k\geq 0}\|g_k\|_{M^1}^2<\infty$ is dense in the set of non-negative operators with kernel in M^1.

Problem Formulation

Interlude: Modulation space M^1

The Feichtinger space M^1 is defined as follows. Let $g : \mathbb{R} \to \mathbb{R}$, $g(x) = e^{-\pi x^2}$ be the Gaussian window. Let

$$f \in \mathbb{S}' \mapsto V_g f(t,w) = \int_{-\infty}^{\infty} e^{-2\pi i w x} f(x) g(x-t) dx$$

be the windowed Fourier transform of f with respect to g. Then

$$M^1(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) \ , \ \|f\|_{M^1} := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |V_g f(t,w)| dt\, dw < \infty \right\}.$$

Problem Formulation

Interlude: Modulation space M^1

The Feichtinger space M^1 is defined as follows. Let $g:\mathbb{R}\to\mathbb{R}$, $g(x)=e^{-\pi x^2}$ be the Gaussian window. Let

$$f\in\mathbb{S}'\mapsto V_gf(t,w)=\int_{-\infty}^{\infty}e^{-2\pi iwx}f(x)g(x-t)dx$$

be the windowed Fourier transform of f with respect to g. Then

$$M^1(\mathbb{R})=\left\{f\in L^2(\mathbb{R})\ ,\ \|f\|_{M^1}:=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|V_gf(t,w)|dt\,dw<\infty\right\}.$$

Fact: [FeichtGrochWaln92] The Wilson ONB is an unconditional basis in M^1. Let $(w_n)_{n\geq 0}$ denote this Wilson basis. Then we can identify M^1 with $l^1(\mathbb{N})$ space, with equivalent norms:

$$M^1(\mathbb{R})=\{f=\sum_{n\geq 0}c_nw_n\ ,\ \|f\|_{M^1}\sim\sum_{n\geq 0}|c_n|\}.$$

Problem (Re)Formulation

Matrix Language

Consider an infinite matrix $A = (A_{m,n})_{m,n\geq 0}$ so that

$$\|A\|_{\wedge} := \|A\|_{1,1} := \sum_{m,n\geq 0} |A_{m,n}| < \infty.$$

This implies that A acts on $l^2(\mathbb{N})$ as a trace-class compact operator.
Assume additionally $A = A^* \geq 0$ as a quadratic form.
Let $(e_k)_{k\geq 0}$ denote an orthogonal set of eigenvectors normalized so that $A = \sum_{k\geq 0} e_k e_k^*$. It is easy to check that $e_k \in l^1(\mathbb{N})$, for each k.
Equivalent reformulations of the two problems:

Problem (Re)Formulation

Matrix Language

Consider an infinite matrix $A = (A_{m,n})_{m,n\geq 0}$ so that

$$\|A\|_{\wedge} := \|A\|_{1,1} := \sum_{m,n\geq 0} |A_{m,n}| < \infty.$$

This implies that A acts on $l^2(\mathbb{N})$ as a trace-class compact operator.
Assume additionally $A = A^* \geq 0$ as a quadratic form.
Let $(e_k)_{k\geq 0}$ denote an orthogonal set of eigenvectors normalized so that $A = \sum_{k\geq 0} e_k e_k^*$. It is easy to check that $e_k \in l^1(\mathbb{N})$, for each k.
Equivalent reformulations of the two problems:
Problem 1: Does it hold $\sum_{k\geq 0} \|e_k\|_1^2 < \infty$?

Problem (Re)Formulation

Matrix Language

Consider an infinite matrix $A = (A_{m,n})_{m,n\geq 0}$ so that

$$\|A\|_{\wedge} := \|A\|_{1,1} := \sum_{m,n\geq 0} |A_{m,n}| < \infty.$$

This implies that A acts on $l^2(\mathbb{N})$ as a trace-class compact operator.
Assume additionally $A = A^* \geq 0$ as a quadratic form.
Let $(e_k)_{k\geq 0}$ denote an orthogonal set of eigenvectors normalized so that $A = \sum_{k\geq 0} e_k e_k^*$. It is easy to check that $e_k \in l^1(\mathbb{N})$, for each k.
Equivalent reformulations of the two problems:
Problem 1: Does it hold $\sum_{k\geq 0} \|e_k\|_1^2 < \infty$?
Problem 2: If negative to problem 1, is there a factorization $A = \sum_{k\geq 0} f_k f_k^*$ so that $\sum_{k\geq 0} \|f_k\|_1^2 < \infty$?

The Good, the Bad ...

Consider the identity matrix I_n and two possible decompositions:

$$I_n = \sum_{k=1}^{n} \delta_k \delta_k^* = \sum_{k=0}^{n-1} e_{n,k} e_{n,k}^*$$

where $\{\delta_k\}_k$ is the canonical ONB, and $\{e_{n,k}\}_k$ is the Fourier ONB:

$$e_{n,k} = \frac{1}{\sqrt{n}} \left[\begin{array}{cccc} 1 & e^{-2\pi ik/n} & \cdots & e^{-2\pi ik(n-1)/n} \end{array} \right]^T.$$

The Good, the Bad ...

Consider the identity matrix I_n and two possible decompositions:

$$I_n = \sum_{k=1}^{n} \delta_k \delta_k^* = \sum_{k=0}^{n-1} e_{n,k} e_{n,k}^*$$

where $\{\delta_k\}_k$ is the canonical ONB, and $\{e_{n,k}\}_k$ is the Fourier ONB:

$$e_{n,k} = \frac{1}{\sqrt{n}} \left[\begin{array}{cccc} 1 & e^{-2\pi i k/n} & \cdots & e^{-2\pi i k(n-1)/n} \end{array} \right]^T.$$

Notice:

$$\sum_{k=1}^{n} \|\delta_k\|_1^2 = n \rightarrow \text{"good decomposition"}$$

$$\sum_{k=0}^{n-1} \|e_{n,k}\|_1^2 = n^2 \rightarrow \text{"bad decomposition"}$$

The (Counter)Example

We construct an example that answers negatively problem 1, but positively problem 2.

Consider the form: $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n \oplus \cdots$,

$$T = \begin{bmatrix} T_1 & & & & \\ & T_2 & & & \\ & & \ddots & & \\ & & & T_n & \\ & & & & \ddots \end{bmatrix}$$

The CounterExample
... and the Ugly

Each block T_n is diagonalized by the Fourier ONB, and has positive simple eigenvalues:

$$T_n = \frac{1}{n^3} \sum_{k=0}^{n-1} \left(1 + \frac{k}{n^p}\right) e_{n,k} e_{n,k}^*.$$

The CounterExample
... and the Ugly

Each block T_n is diagonalized by the Fourier ONB, and has positive simple eigenvalues:

$$T_n = \frac{1}{n^3}\sum_{k=0}^{n-1}\left(1+\frac{k}{n^p}\right)e_{n,k}e_{n,k}^*.$$

Thus:

$$T = \bigoplus_{n\geq 1}\sum_{k=0}^{n-1}\frac{1}{n^3}\left(1+\frac{k}{n^p}\right)e_{n,k}e_{n,k}^*.$$

Problem 1
Negative Answer

The eigendecomposition of T is

$$T = \sum_{n\geq 1} \sum_{k=0}^{n-1} f_{n,k} f_{n,k}^* \quad , \quad f_{n,k} = \frac{1}{\sqrt{n^3}} \sqrt{1 + \frac{k}{n^p}} e_{n,k}.$$

Then

$$\sum_{n\geq 1} \sum_{k=0}^{n-1} \|f_{n,k}\|_1^2 = \sum_{n\geq 1} \sum_{k=0}^{n-1} \frac{1}{n^3}(1 + \frac{k}{n^p})n \geq \sum_{n\geq 1} \frac{1}{n} = \infty$$

Hence the answer to problem 1 is negative: There is an operator $S : f \mapsto Sf(x) = \int K(x,y)f(y)dy$ with $K \in M^1(\mathbb{R}^2)$ and $S = S^* \geq 0$, so that its spectral decomposition $S = \sum_{k\geq 1} \langle \cdot, f_k \rangle f_k$ satisfies $\sum_k \|f_k\|_{M^1}^2 = \infty$.

Problem 2
Positive Answer

We show now that same operator T we constructed earlier admits a decomposition $T = \sum_m g_m g_m^*$ so that $\sum_m \|g_m\|_1^2 < \infty$.
Notice:

$$T_n = \frac{1}{n^3} \sum_{k=0}^{n-1} \left(1 + \frac{k}{n^p}\right) e_{n,k} e_{n,k}^* = \frac{1}{n^3} \sum_{k=0}^{n-1} \delta_k \delta_k^* + \frac{1}{n^{3+p}} \sum_{k=0}^{n-1} k e_{n,k} e_{n,k}^*$$

Thus the induced decomposition

$$T_n = \sum_{k=0}^{n-1} g_{1,n,k} g_{1,n,k}^* + \sum_{k=0}^{n-1} g_{2,n,k} g_{2,n,k}^*$$

satisfies

$$\sum_{k=0}^{n-1} \|g_{1,n,k}\|_1^2 + \|g_{2,n,k}\|_1^2 = \frac{1}{n^2} + \frac{1}{n^{2+p}} \frac{n(n-1)}{2} \leq \frac{1}{n^2} + \frac{1}{n^p}$$

Problem 2
Positive Answer - cont'd

Thus:

$$T = \bigoplus_{n\geq 1} \sum_{k=0}^{n-1} g_{1,n,k} g_{1,n,k}^{*} + g_{2,n,k} g_{2,n,k}^{*}$$

satisfies

$$\sum_{n\geq 1} \sum_{k=0}^{n-1} \|g_{1,n,k}\|_1^2 + \|g_{2,n,k}\|_1^2 \leq \sum_{n\geq 1} \frac{1}{n^2} + \frac{1}{n^p} < \infty$$

Problem 2

Positive Answer - cont'd

Thus:

$$T = \bigoplus_{n\geq 1} \sum_{k=0}^{n-1} g_{1,n,k} g_{1,n,k}^* + g_{2,n,k} g_{2,n,k}^*$$

satisfies

$$\sum_{n\geq 1} \sum_{k=0}^{n-1} \|g_{1,n,k}\|_1^2 + \|g_{2,n,k}\|_1^2 \leq \sum_{n\geq 1} \frac{1}{n^2} + \frac{1}{n^p} < \infty$$

Hence the answer to the second problem is affirmative: There is an operator $S = S^* \geq 0$, $f \mapsto Sf(x) = \int K(x,y)f(y)dy$ with $K \in M^1(\mathbb{R}^2)$ that admits a decomposition $S = \sum_{k\geq 1} \langle \cdot, g_k \rangle g_k$ that satisfies $\sum_k \|g_k\|_{M^1}^2 < \infty$, but whose spectral decomposition does not satisfy the same localization condition.

Tensor Products

Consider $A \in \mathbb{C}^{n \times n}$. We seek "optimal" decompositions of A into a sum of rank-1 operators: $A = \sum_k u_k v_k^*$.
In this talk we assume A to be positive semi-definite: $A = A^* \geq 0$.
Criterion 1:

$$J_+(A) = \inf_{A=\sum_{k=1}^{m} f_k f_k^*} \sum_{k=1}^{m} \|f_k\|_1^2.$$

Tensor Products

Consider $A \in \mathbb{C}^{n\times n}$. We seek "optimal" decompositions of A into a sum of rank-1 operators: $A = \sum_k u_k v_k^*$.
In this talk we assume A to be positive semi-definite: $A = A^* \geq 0$.

Criterion 1:

$$J_+(A) = \inf_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2.$$

Criterion 2:

$$J_0(A) = \inf_{A=\sum_{k=1}^m \epsilon_k f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2$$

where $\epsilon_k \in \{+1, -1\}$.

Tensor Products

Consider $A \in \mathbb{C}^{n\times n}$. We seek "optimal" decompositions of A into a sum of rank-1 operators: $A = \sum_k u_k v_k^*$.

In this talk we assume A to be positive semi-definite: $A = A^* \geq 0$.

Criterion 1:

$$J_+(A) = \inf_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2.$$

Criterion 2:

$$J_0(A) = \inf_{A=\sum_{k=1}^m \epsilon_k f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2$$

where $\epsilon_k \in \{+1, -1\}$.

Criterion 3:

$$J(A) = \inf_{A=\sum_{k=1}^m f_k g_k^*} \sum_{k=1}^m \|f_k\|_1 \|g_k\|_1$$

What we know

$$J(A) = \inf_{A=\sum_{k=1}^{m} f_k g_k^*} \sum_{k=1}^{m} \|f_k\|_1 \|g_k\|_1$$

$$J_0(A) = \inf_{A=\sum_{k=1}^{m} \epsilon_k f_k f_k^*} \sum_{k=1}^{m} \|f_k\|_1^2$$

$$J_+(A) = \inf_{A=\sum_{k=1}^{m} f_k f_k^*} \sum_{k=1}^{m} \|f_k\|_1^2.$$

What we know

$$J(A) = \inf_{A=\sum_{k=1}^m f_k g_k^*} \sum_{k=1}^m \|f_k\|_1 \|g_k\|_1$$

$$J_0(A) = \inf_{A=\sum_{k=1}^m \epsilon_k f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2$$

$$J_+(A) = \inf_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2.$$

1. $J_\wedge, J_0, J$ are positive, homogeneous, and convex on $Sym^+(\mathbb{C}^n)$.

What we know

$$J(A) = \inf_{A=\sum_{k=1}^m f_k g_k^*} \sum_{k=1}^m \|f_k\|_1 \|g_k\|_1$$

$$J_0(A) = \inf_{A=\sum_{k=1}^m \epsilon_k f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2$$

$$J_+(A) = \inf_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2.$$

1. $J_\wedge, J_0, J$ are positive, homogeneous, and convex on $Sym^+(\mathbb{C}^n)$.
2. J, J_0 extend to norms on $Sym(\mathbb{C}^n)$.

What we know

$$J(A) = \inf_{A=\sum_{k=1}^m f_k g_k^*} \sum_{k=1}^m \|f_k\|_1 \|g_k\|_1$$

$$J_0(A) = \inf_{A=\sum_{k=1}^m \epsilon_k f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2$$

$$J_+(A) = \inf_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2.$$

1. $J_\wedge, J_0, J$ are positive, homogeneous, and convex on $Sym^+(\mathbb{C}^n)$.
2. J, J_0 extend to norms on $Sym(\mathbb{C}^n)$.
3. The following hold true:

$$\sum_{i,j} |A_{i,j}| =: \|A\|_{1,1} = J \leq J_0(A) \leq 2\|A\|_{1,1} \quad , \quad \forall A \in Sym(\mathbb{C}^n).$$

$$\|A\|_{1,1} = J \leq J_0(A) \leq J_+(A) \leq n\|A\|_{1,1} \quad , \quad \forall A \in Sym^+(\mathbb{C}^n).$$

Hypothesis

We posit the following hypothesis: There is a universal constant $C_0 < \infty$ so that for any $n \geq 1$ and every positive semidefinite $A \in \mathbb{C}^{n\times n}$,

$$J_+(A) = \inf_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2 \leq C_0 \sum_{i,j=1}^n |A_{i,j}| \quad (H)$$

Hypothesis

We posit the following hypothesis: There is a universal constant $C_0 < \infty$ so that for any $n \geq 1$ and every positive semidefinite $A \in \mathbb{C}^{n \times n}$,

$$J_+(A) = \inf_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2 \leq C_0 \sum_{i,j=1}^n |A_{i,j}| \quad (H)$$

In a different formulation: The sequence $(C_n)_{n\geq 1}$,

$$C_n = \sup_{A \in S^+(\mathbb{C}^n)\ :\ \|A\|_{1,1}=1} \inf_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2$$

is bounded.

Notice the sequence is monotonically increasing, $C_n \leq C_{n+1}$ by a simple bordering argument. Hence the hypothesis is equivalent to:

$$\lim_{n\to\infty} C_n = C_0 < \infty \quad (H)$$

Consequences of the Hypothesis

If the Hypothesis is False

Theorem (A)

If Hypothesis (H) is false, then there exists an operator $A \in Sym^+(l^2(\mathbb{N}))$ *with* $\|A\|_{1,1} < \infty$ *so that for any operator-norm convergent expansion* $A = \sum_{k\geq 1} f_k f_k^*$, *the series* $\sum_{k\geq 1} \|f_k\|_1^2 = \infty$ *is divergent .*

In the T-F language:

Theorem (B)

If Hypothesis (H) is false, then there is a positive trace-class operator $T \in Sym^+(L^2(\mathbb{R}))$ *with kernel* $K \in M^1(\mathbb{R}^2)$ *so that for any operator-norm convergent expansion* $T = \sum_{k\geq 1} \langle \cdot, f_k \rangle f_k$, *the series* $\sum_{k\geq 1} \|f_k\|_{M^1}^2 = \infty$ *is divergent.*

If the Hypothesis is False
Proof of Theorem A

Proof of Theorem A:
For each $n = 1, 2, ...$ let $A_n \in Sym^+(\mathbb{C}^n)$ so that $\|A_n\|_{1,1} = 1$, $C_n = J_+(A_n)$ and $\lim_{n\to\infty} J_+(A_n) = \infty$. Let $(w_n)_{n\geq 1}$ be a sequence of non-negative numbers so that $\sum_{n\geq 1} w_n < \infty$ but $\sum_{n\geq 1} w_n C_n = \infty$. Then consider the operator

$$A = (w_1 A_1) \oplus (w_2 A_2) \oplus \cdots \oplus (w_n A_n) \oplus \cdots$$

acting on $l^2(\mathbb{N})$. A direct computation shows $A \in Sym^+(l^2(\mathbb{N}))$ and $\|A\|_{1,1} = \sum_{n\geq 1} w_n < \infty$. On the other hand, let $A = \sum_{k\geq 1} f_k f_k^*$ a decomposition of A into rank-1 matrices and let $P_1, P_2, \cdots, P_n, \cdots$ the orthogonal projections onto the corresponding block in matrix A. Thus $PAP = 0 \oplus \cdots \oplus 0 \oplus A_n \oplus 0 \oplus \cdots$ and $P_1 + P_2 + \cdots + P_n + \cdots = 1$.

If the Hypothesis is False
Proof of Theorem A - cont'd

Let $f_{k,n} = P_n f_k$. Then

$$A = \sum_{n,m\geq 1} \sum_{k\geq 1} f_{k,n} f_{k,m}^* = \sum_{n\geq 1} \sum_{k\geq 1} f_{n,k} f_{n,k}^*$$

because the off-diagonal blocks must vanish. But then $\sum_{k\geq 1} \|f_k\|_1^2 \geq \sum_{n\geq 1} \sum_{k\geq 1} \|f_{n,k}\|_1^2$ which implies that the optimal decomposition of A involves expansions of each block A_n independently. Therefore

$$J_+(A) = \sum_{n\geq 1} J_+(A_n) = \sum_{n\geq 1} w_n C_n = \infty.$$

This shows Theorem A.

Theorem B is an immediate consequence.

Consequences of the Hypothesis

If the Hypothesis is True

Theorem (C)

If the hypothesis (H) is true, then for any operator $A \in Sym^{+}(l^2(\mathbb{N}))$ *with* $\|A\|_{1,1} < \infty$, *and any* $\varepsilon > 0$ *there are vectors* $f_k, g_k \in l^1(\mathbb{N})$, $k = 1, 2, \ldots$, *so that the operator-norm convergent expansion* $A = \sum_{k \geq 1} f_k f_k^* - \sum_{k \geq 1} g_k g_k^*$ *satisfies*

$$\sum_{k \geq 1} \|f_k\|_1^2 \leq C_0 \|A\|_{1,1} + \varepsilon \quad , \quad \sum_{k \geq 1} \|g_k\|_1^2 < \varepsilon.$$

In particular, the set

$$\mathbb{S} = \{A \in Sym^{+}(l^2(\mathbb{N})), \|A\|_{1,1} < \infty, \exists (f_k)_k : A = \sum_{k \geq 1} f_k f_k^*, \sum_{k \geq 1} \|f_k\|_1^2 < \infty\}$$

is dense in $\{A \in Sym^{+}(l^2(\mathbb{N})) \ , \ \|A\|_{1,1} < \infty\}$.

Consequences of the Hypothesis

If the Hypothesis is True

Theorem (D)

If the hypothesis (H) is true, then for any operator $T \in Sym^+(L^2(\mathbb{R}))$ *with kernel* $K \in M^1(\mathbb{R}^2)$*, and any* $\varepsilon > 0$ *there are vectors* $f_k, g_k \in M^1(\mathbb{R})$*,* $k = 1, 2, \ldots$*, so that the operator-norm convergent expansion* $T = \sum_{k\geq 1} \langle \cdot, f_k \rangle f_k - \sum_{k\geq 1} \langle \cdot, g_k \rangle g_k$ *satisfies*

$$\sum_{k\geq 1} \|f_k\|^2_{M^1} \leq C_0 \|K\|_{M^1(\mathbb{R}^2)} + \varepsilon \quad , \quad \sum_{k\geq 1} \|g_k\|^2_{M^1} < \varepsilon.$$

In particular, the set

$$\mathbb{S} = \{T \in Sym^+(L^2(\mathbb{R})), \|K\|_{M^1(\mathbb{R}^2)} < \infty, \exists (f_k)_k : \; A = \sum_{k\geq 1} \langle \cdot, f_k \rangle f_k, \sum_{k\geq 1} \|f_k\|^2_{M^1} < \infty\}$$

is dense in $\{T \in Sym^+(L^2(\mathbb{R})) \; , \; K \in M^1(\mathbb{R}^2)\}$.

If the Hypothesis is True
Proof of Theorem C

Proof of Theorem C:
Fix $A = A^* \geq 0$ with $\|A\|_{1,1} < \infty$, and $\varepsilon > 0$. Let n be large enough so that the central $[0,n] \times [0,n]$ block A_n of A carries the norm within ε / C_0: $\|A\|_{1,1} \geq \sum_{0 \leq k,j \leq n} |A_{k,j}| > \|A\|_{1,1} - \frac{\varepsilon}{C_0}$. Then let $f_1, \cdots, f_m$ be a decomposition of A_n,

$$A_n = \sum_{k=1}^{m} f_k f_k^* \quad \text{so that } \|f_k\|_1^2 \leq C_0 \|A_n\|_{1,1} \leq C_0 \|A\|_{1,1}.$$

Let $B = A - A_n \in Sym(l^2(\mathbb{N}))$ be the residual operator. Using the fact that $J_0(B) \leq 2\|B\|_{1,1} < \frac{2\varepsilon}{C_0} \leq \varepsilon$ let $f_{m+1}, f_{m+1}, \cdots, g_1, g_2, \cdots \in l^1(\mathbb{N})$ be so that:

$$B = \sum_{k \geq m+1} f_k f_k^* - \sum_{k \geq 1} g_k g_k^*$$

and

$$\sum_{k \geq m+1} \|f_k\|_1^2 + \sum_{k \geq 1} \|g_k\|_1^2 \leq \varepsilon.$$

If the Hypothesis is True

Proof of Theorem C

Putting together the two expansions, it follows

$$A = \sum_{k\geq 1} f_k f_k^* - \sum_{k\geq 1} g_k g_k^* \quad , \quad \sum_{k\geq 1} \|f_k\|_1^2 \leq C_0 \|A\|_{1,1} + \varepsilon \ , \ \sum_{k\geq 1} \|g_k\|_1^2 < \varepsilon.$$

Theorem D follows similarly.

THANK YOU!!

...

QUESTIONS?

References

R. Balan, K. Okoudjou, A. Poria, *On a Feichtinger Problem*, available online at arXiv:1705.06392 [math.CA], Oper. Matrices, 12 (2018), no. 3, 881–891.

I. Daubechies, S. Jaffard, and J.-L. Journé, *A simple Wilson orthonormal basis with exponential decay*, SIAM J. Math. Anal., **22** (1991), 554–573.

N. Dunford and J. T. Schwartz, "Linear operators, Part II", Wiley, New York, 1988.

H. Feichtinger, P. Jorgensen, D. Larson and G. Ólafsson, *Mini-Workshop: Wavelets and Frames*, Abstracts from the mini-workshop held February 15-21, 2004, Oberwolfach Rep. **1** (2004), no. 1, 479–543.

H. G. Feichtinger, *Modulation spaces on locally compact Abelian groups*, in: Wavelets and their Applications (Chennai, January 2002),

M. Krishna, R. Radha and S. Thangavelu, eds., Allied Publishers, New Delhi (2003), pp. 1–56.

H. G. Feichtinger, K. Gröchenig, and D. Walnut, *Wilson bases and modulation spaces*, Math. Nachr., **155** (1992), 7–17.

K. Gröchenig, "Foundations of time-frequency analysis", Birkhäuser, Boston, 2001.

K. Gröchenig and C. Heil, *Modulation spaces and pseudodifferential operators*, Integr. Equ. Oper. Theory, **34** (1999), 439–457.

C. Heil and D. Larson, *Operator theory and modulation spaces*, Contemp. Math., **451** (2008), 137–150.

B. Simon, "Trace ideals and their applications", Cambridge University Press, Cambridge, 1979.

K. G. Wilson, *Generalized Wannier functions*, preprint (1987).