Global Lipschitz Analysis in Inverse Problems

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Joint work with: Yang Wang (HKST), Dongmian Zou (IMA).

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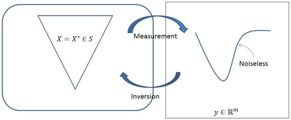
High-Level Problem Formulation

Measured: We are given a set of measurements $y = (y_k)_k$ associated to a positive semidefinite matrix $X = X^* \geq 0$.
Unknown: We want to estimate/reconstruct the operator X from these measurements.



Today problem: Determine fundamental limits to robustness and stability of any inversion algorithm.

Quantum Tomography
Problem

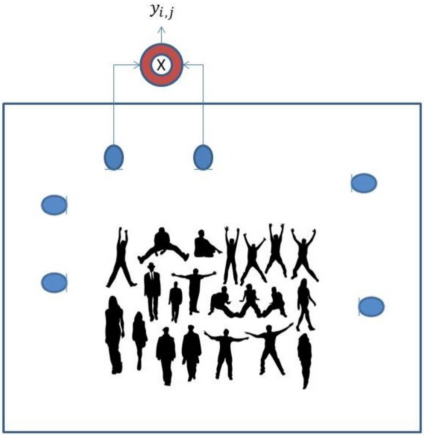
Given a quantum system in the (mixed) quantum state $M \in \mathbb{C}^{n \times n}$, and a set of observables $Y_1, \cdots, Y_m$ that can be measured simultaneously, assume we know

$$y_k = trace(MY_k) \quad , \quad 1 \leq k \leq m.$$

The problem is to estimate (compute) the PSD $M = M^* \geq 0$ that satisfies $trace(M) = 1$.
Additionally we assume M has low rank, for instance $rank(M) \leq d$.

Scene Understanding from Power Measurements

Problem



Mixing model: n decorrelated sources (acoustic, RF, etc) monitored by n sensors. A subset S of all possible ordered pairs $\{(i,j)\,;\,1 \leq i \leq j \leq n\}$ of sensors determines signal covariance, i.e. the measurements are:

$$y_\alpha = \mathbb{E}[x_i \overline{x_j}] + \nu_\alpha = R_{i,j} + \nu_\alpha.$$

for $\alpha = (i,j) \in S$ and $R = \mathbb{E}[xx^*]$ is the $n \times n$ cov. matrix of rank d.

The problem is to estimate R from the set of measurements $\{y_\alpha\ ,\ \alpha \in S\}$ $(|S| = m)$.

Setup

Notations

$H = \mathbb{R}^n$ or $H = \mathbb{C}^n$, finite dimensional Euclidean space.

- $Sym(\mathbb{R}^n) = \{T \in \mathbb{R}^{n\times n}\ ,\ T = T^T\}$ or
  $Sym(\mathbb{C}^n) = \{T \in \mathbb{C}^{n\times n}\ ,\ T = T^*\}$
- Convex cone of PSD: $Sym^+(H) = \{T \in Sym(H)\ ,\ T = T^* \geq 0\}$

Setup
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- Convex cone of PSD: $Sym^+(H) = \{T \in Sym(H)\ ,\ T = T^* \geq 0\}$
- Quantum states: $St(H) = \{T \in Sym^+(H)\ ,\ trace(T) = 1\}$
- Cone of mixed signatures matrices:

  $$\mathcal{S}^{p,q}\{T \in Sym(H)\ ,\ T \text{ has at most } p \text{ positive and } q \text{ negative eigenvalu}$$

  In particular $\mathcal{S}^{1,0} = \{xx^*\ ,\ x \in H\}$, set of rank (at most) one PSDs.
- Low-rank quantum states $St^r(H) = \{T \in Sym^+(H)\ ,\ trace(T) = 1\ ,\ rank(T) \leq r\}$

Problem Formulation
Models

Measurement maps:

$$\alpha : Sym^+(H) \to \mathbb{R}^m \quad , \quad (\alpha(X))_k = \sqrt{trace(XF_k)}$$

$$\beta : Sym^+(H) \to \mathbb{R}^m \quad , \quad (\beta(X))_k = trace(XF_k)$$

where $F_1, \cdots, F_m \in Sym^+(H)$ are fixed PSD matrices.

Problem Formulation

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where $F_1, \cdots, F_m \in Sym^{+}(H)$ are fixed PSD matrices.

Prior Information: Assume $X \in \mathcal{S} \subset Sym^{+}(H)$:

- Phase Retrieval: $\mathcal{S} = \mathcal{S}^{1,0} = \{xx^* \ , \ x \in H\}$.
- Quantum Tomography:
  $\mathcal{S} = St^r(H) = \{X = X^* \geq 0 \ , \ trace(X) = 1 \ , \ rank(X) \leq r\}$.
- Covariance Estimation: $\mathcal{S} = \mathcal{S}^{d,0}$.

Problem Formulation
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- Covariance Estimation: $\mathcal{S} = \mathcal{S}^{d,0}$.

Matrix Estimation Problem: Estimate $X \in \mathcal{S}$ given $y = \alpha(X) + \nu$ or $y = \beta(X) + \nu$.

Problem Formulation

The phase retrieval problem

Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H/T^1$, frame $\mathcal{F} = \{f_1, \cdots, f_m\} \subset \mathbb{C}^n$ and

$$\alpha : \hat{H} \to \mathbb{R}^m \quad , \quad \alpha(x) = (|\langle x, f_k \rangle|)_{1 \leq k \leq m}.$$

$$\beta : \hat{H} \to \mathbb{R}^m \quad , \quad \beta(x) = \left(|\langle x, f_k \rangle|^2\right)_{1 \leq k \leq m}.$$

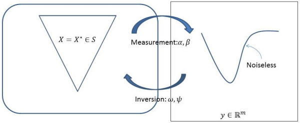
Assume α, β are injective, the problem is to construct global Lipschitz inverses and to study their constants.

Problem Formulation

Lipschitz reconstruction: the general case

Assume the maps $\alpha : \mathcal{S} \to \mathbb{R}^m$, $X \mapsto (\alpha(X)) = \sqrt{trace(XF_k)}$ and $\beta : \mathcal{S} \to \mathbb{R}^m$, $X \mapsto (\beta(X))_k = trace(XF_k)$ are injective.

Our Problem Today:



Construct Lipschitz maps $\psi, \omega : \mathbb{R}^m \to \mathcal{S}$ so that for every $X \in \mathcal{S}$,

$$\omega(\alpha(X)) = X = \psi(\beta(X)).$$

Determine $Lip(\psi)$ and $Lip(\omega)$.

Metric Structures on $\hat{H}$ and $Sym(H)$

Norm Induced Metric

Fix $1 \leq p \leq \infty$. The *matrix-norm induced distance* on $Sym(H)$:

$$d_p : Sym(H) \times Sym(H) \to \mathbb{R} \ , \ d_p(X, Y) = \|X - Y\|_p,$$

the p-norm of the singular values.
On $\hat{H} = H/T^1$ it induces the metric

$$\mathbf{d}_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \mathbf{d}_p(\hat{x}, \hat{y}) = \|xx^* - yy^*\|_p$$

so that $\mathbf{d}_p(\hat{x}, \hat{y}) = d_p(xx^*, yy^*)$. In the case $p = 2$ we obtain

$$d_2(X, Y) = \|X - Y\|_F^2 \quad , \quad \mathbf{d}_2(x, y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x, y\rangle|^2}$$

Metric Structures on $\hat{H}$ and $Sym(H)$

The Natural Metric

The *natural metric*

$$\mathbf{D}_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \ \mathbf{D}_p(\hat{x}, \hat{y}) = \min_{\varphi} \|x - e^{i\varphi} y\|_p$$

with the usual p-norm on $\mathbb{C}^n$. In the case $p = 2$ we obtain

$$\mathbf{D}_2(\hat{x}, \hat{y}) = \sqrt{\|x\|^2 + \|y\|^2 - 2|\langle x, y\rangle|}$$

On $Sym^+(H)$, the "natural" metric lifts to

$$D_p : Sym^+(H) \times Sym^+(H) \to \mathbb{R} \ , \ \ D_p(X, Y) = \min_{\substack{VV^* = X \\ WW^* = Y}} \|V - W\|_p.$$

Metric Structures on $Sym(H)$

Natural metric vs. Bures/Helinger

Let $X, Y \in Sym^+(H)$. For the natural distance we choose $p = 2$:

$$D_{natural}(X,Y) = \min_{\substack{VV^* = X \\ WW^* = Y}} \|V - W\|_F$$

Fact (easy):

$$D_{natural}(X,Y) = \min_{U \in U(n)} \|X^{1/2} - Y^{1/2}U\|_F = \sqrt{\mathrm{tr}(X) + tr(Y) - 2\|X^{1/2}Y^{1/2}\|_1}$$

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Another distance: Bures/Helinger distance:

$$D_{Bures}(X, Y) = \|X^{1/2} - Y^{1/2}\|_F = d_2(X^{1/2}, Y^{1/2})$$

Metric Structures on $Sym(H)$

Natural metric vs. Bures/Helinger

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A consequence of the Arithmetic-Geometric Mean Inequality [BhatiaKittaneh00]:

$$\frac{1}{2} D^2_{Bures}(X,Y) \le D^2_{natural}(X,Y) \le D^2_{Bures}(X,Y).$$

Stability Results for the forward maps

Bi-Lipschitz properties of α and β

Fix a closed subset $S \subset Sym^+(H)$. For instance $S = St(H)$, or $St^r(H)$, or $\mathcal{S}^{r,0}$.

Theorem

Assume $\mathcal{F} = \{F_1, \cdots, F_m\} \subset Sym^+(H)$ so that $\alpha|_S$ and $\beta|_S$ are injective. Then there are constants $a_0, A_0, b_0, B_0 > 0$ so that for every $X, Y \in S$,

$$A_0 \|X^{1/2} - Y^{1/2}\|_F^2 \leq \sum_{k=1}^{m} \left| \sqrt{\langle X, F_k \rangle} - \sqrt{\langle Y, F_k \rangle} \right|^2 \leq B_0 \|X^{1/2} - Y^{1/2}\|_F^2$$

$$a_0 \|X - Y\|_F^2 \leq \sum_{k=1}^{m} |\langle X, F_k \rangle - \langle Y, F_k \rangle|^2 \leq b_0 \|X - Y\|_F^2.$$

Stability Results for the inverse map

Lipschitz inversion of α and β on Quantum States

Consider the measurement map

$$\beta : (St^r(H), d_1) \rightarrow (\mathbb{R}^m, \|\cdot\|_2) \quad , \quad \beta(T) = (tr(TF_k))_{1 \leq k \leq m}$$

where $St^r(H) = \{T = T^* \geq 0 \ , \ tr(T) = 1 \ , \ rank(T) \leq r\}$.
If $r = n := dim(H)$ then $St^n(H) = St(H)$ is a compact convex set, hence a Lipschitz retract.
Conjecture: If $r < n$ then $St^r(H)$ is not contractible hence not a Lipschitz retract (true for $r = 1$).

Stability Results for the inverse map

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If $r = n := dim(H)$ then $St^n(H) = St(H)$ is a compact convex set, hence a Lipschitz retract.
Conjecture: If $r < n$ then $St^r(H)$ is not contractible hence not a Lipschitz retract (true for $r = 1$). Consequence:
Even if β is injective on rank-r quantum states, there is no globally Lipschitz left inverse map. Same result for the α map.

Stability Results for the inverse map

Lipschitz inversion of α and β on Low-Rank PSD

Theorem

Assume the map

$$\alpha : (\mathcal{S}^{r,0}(H), D_2) \rightarrow (\mathbb{R}^m, \|\cdot\|_2) \quad , \quad (\alpha(T))_k = \sqrt{trace(TF_k)}$$

is injective, where $\mathcal{S}^{r,0}(H) = \{T = T^* \geq 0\ ,\ ,\ rank(T) \leq r\}$. *Then there exists a Lipschitz map* $\omega : \mathbb{R}^m \rightarrow \mathcal{S}$ *so that* $\omega(\alpha(T)) = T$ *for every* $T \in \mathcal{S}$.

Theorem

Assume the map

$$\beta : (\mathcal{S}^{r,0}(H), d_1) \rightarrow (\mathbb{R}^m, \|\cdot\|_2) \quad , \quad (\beta(T))_k = trace(TF_k)$$

is injective, where $\mathcal{S}^{r,0}(H) = \{T = T^* \geq 0\ ,\ ,\ rank(T) \leq r\}$. *Then there exists a Lipschitz map* $\psi : \mathbb{R}^m \rightarrow \mathcal{S}$ *so that* $\psi(\beta(T)) = T$ *for every* $T \in \mathcal{S}$.

Phase Retrieval

Lipschitz inversion: α

Theorem (BZ15)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\alpha : (\hat{H}, \mathbf{D}_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{A_0}, \sqrt{B_0}$ denote its Lipschitz constants: for every $x, y \in H$:*

$$A_0 \min_{\varphi} \|x - e^{i\varphi}y\|_2^2 \leq \sum_{k=1}^{m} \left||\langle x, f_k\rangle| - |\langle y, f_k\rangle|\right|^2 \leq B_0 \min_{\varphi} \|x - e^{i\varphi}y\|_2^2.$$

2. *There is a Lipschitz map $\omega : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, D_2)$ so that: (i) $\omega(\alpha(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $Lip(\omega) \leq \frac{4+3\sqrt{2}}{\sqrt{A_0}} = \frac{8.24}{\sqrt{A_0}}$.*

Phase Retrieval

Lipschitz inversion: β

Theorem (BZ15)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\beta : (\hat{H}, \mathbf{d}_1) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{a_0}, \sqrt{b_0}$ denote its Lipschitz constants: for every $x, y \in H$:*

$$a_0 \|xx^* - yy^*\|_1^2 \leq \sum_{k=1}^{m} \left| |\langle x, f_k \rangle|^2 - |\langle y, f_k \rangle|^2 \right|^2 \leq b_0 \|xx^* - yy^*\|_1^2.$$

2. *There is a Lipschitz map $\psi : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, d_1)$ so that: (i) $\psi(\beta(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $Lip(\psi) \leq \frac{4+3\sqrt{2}}{\sqrt{a_0}} = \frac{8.24}{\sqrt{a_0}}$.*

Proofs
Overview

Phase Retrieval: The proofs involve several steps (details in [BZ15]).

1. Part 1: Injectivity $\longrightarrow$ bi-Lipschitz: Upper bounds are not too hard; lower bounds: relatively easy for β (the "square" map), but relatively hard for α.
2. Part 2: Left inverse construction is done in three steps:
    1. The left inverse is first extended to $\mathbb{R}^m$ into $Sym(H)$ using Kirszbraun's theorem;
    2. Then we show that $\mathcal{S}^{1,0}(H)$ is a Lipschitz retract in $Sym(H)$;
    3. The proof is concluded by composing the two maps.

The Low-Rank PSD Case: Similar to the PR case; different Lipschitz retraction for $\mathcal{S}^{r,0}(H)$.

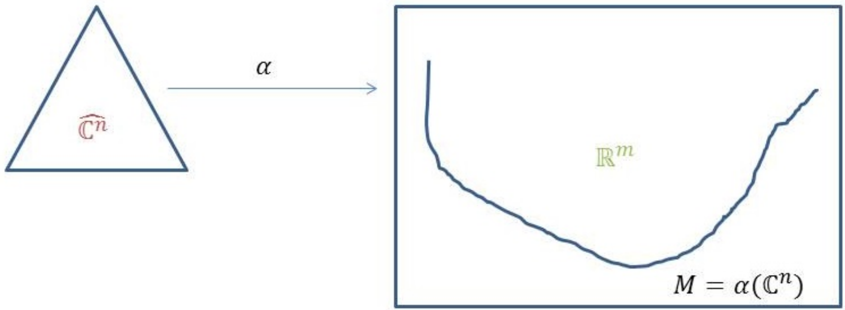
Proofs

Part 2a: Extension of the inverse for α

We know $\alpha : (\hat{H}, \mathbf{D}_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz:

$$A_0 \mathbf{D}_2(x,y)^2 \leq \|\alpha(x) - \alpha(y)\|^2 \leq b_0 \mathbf{D}_2(x,y)^2$$

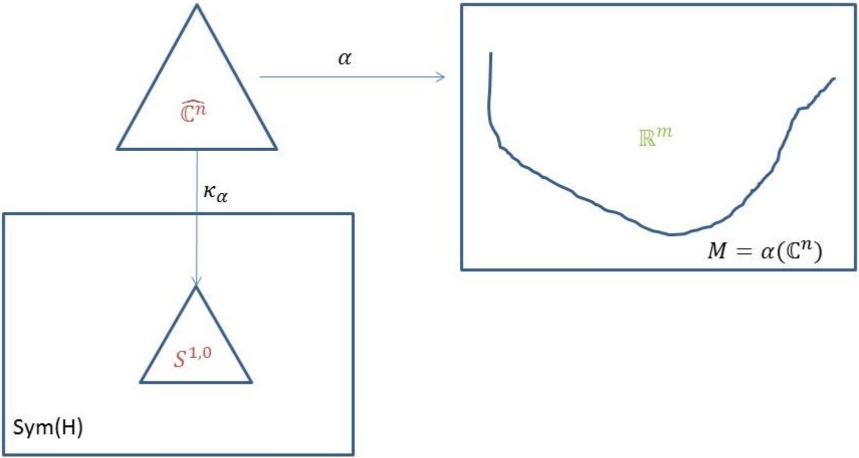
Let $M = \alpha(\hat{H}) \subset \mathbb{R}^m$.



Proofs

Part 2a: Extension of the inverse for α

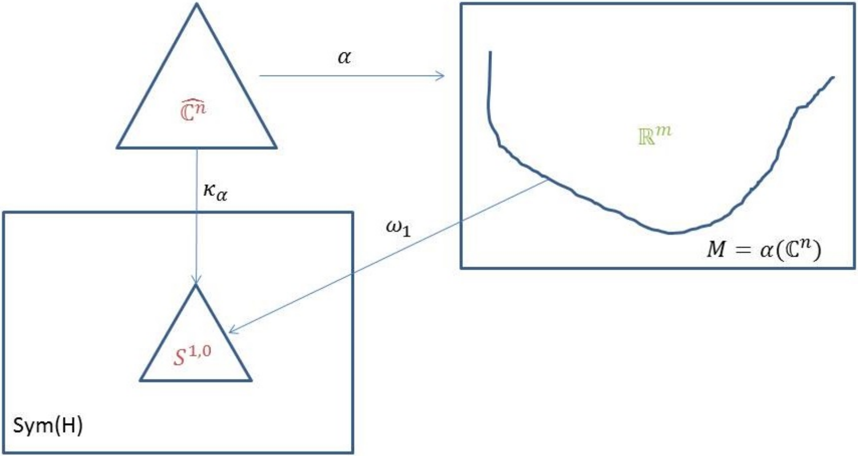
First identify $\hat{H}$ with $\mathcal{S}^{1,0}(H)$.



Proofs

Part 2a: Extension of the inverse for α

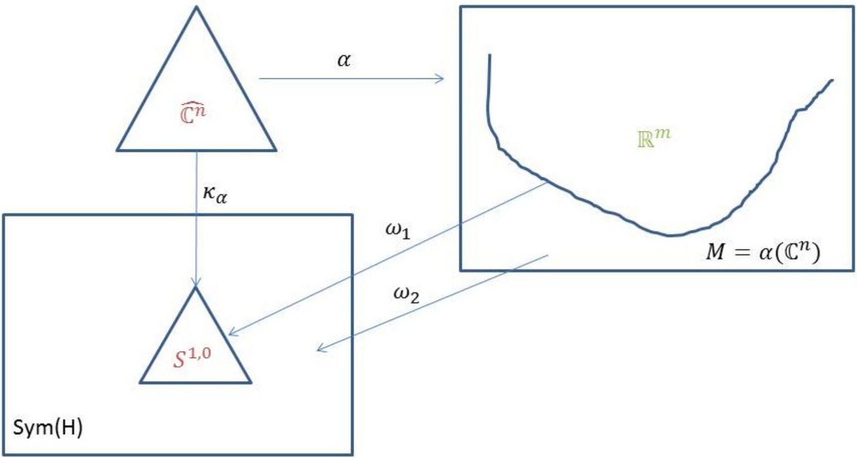
Then construct the local left inverse $\omega_1 : M \to \hat{H}$ with $Lip(\omega_1) = \frac{1}{\sqrt{A_0}}$.



Proofs

Part 2a: Extension of the inverse for α

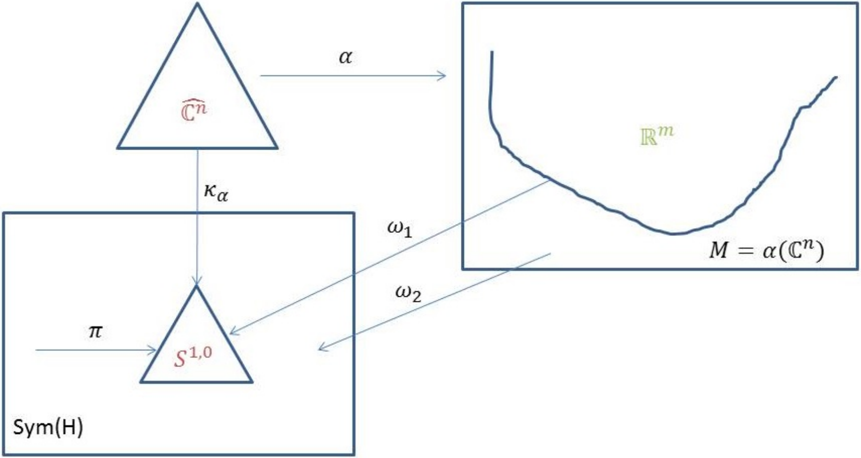
Use Kirszbraun's theorem to extend isometrically $\omega_2 : \mathbb{R}^m \to Sym(H)$.



Proofs

Part 2a: Extension of the inverse for α

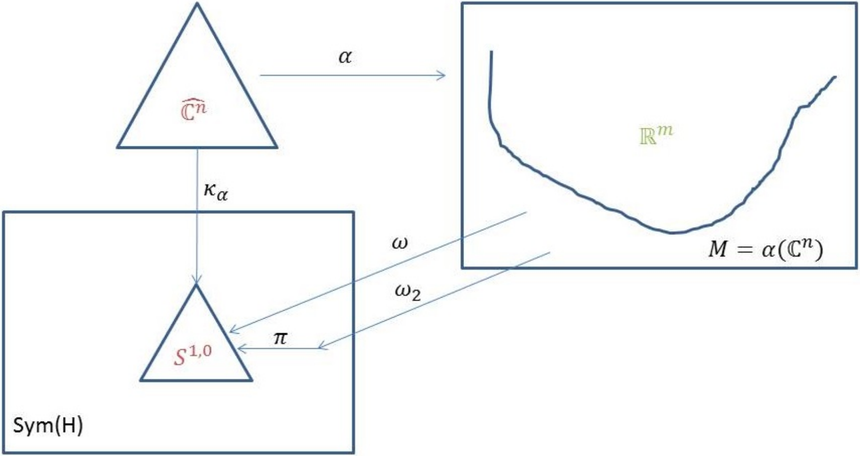
Construct a Lipschitz "projection" $\pi : Sym(H) \rightarrow \mathcal{S}^{1,0}(H)$.



Proofs

Part 2a: Extension of the inverse for α

Compose the two maps to get $\omega : \mathbb{R}^m \to \mathcal{S}^{1,0}$, $\omega = \pi \circ \omega_2$.



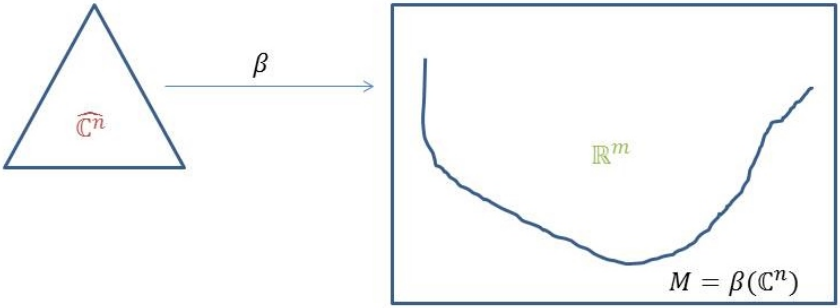
Proofs

Part 2b: Extension of the inverse for β

We know $\beta : (\hat{H}, d_1) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz:

$$a_0 d_1(x,y)^2 \leq \|\beta(x) - \beta(y)\|^2 \leq b_0 d_1(x,y)^2.$$

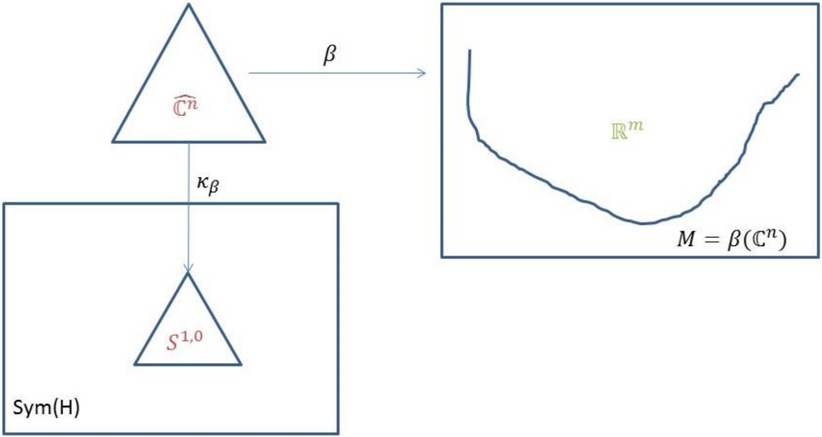
Let $M = \beta(\hat{H}) \subset \mathbb{R}^m$.



Proofs

Part 2b: Extension of the inverse for β

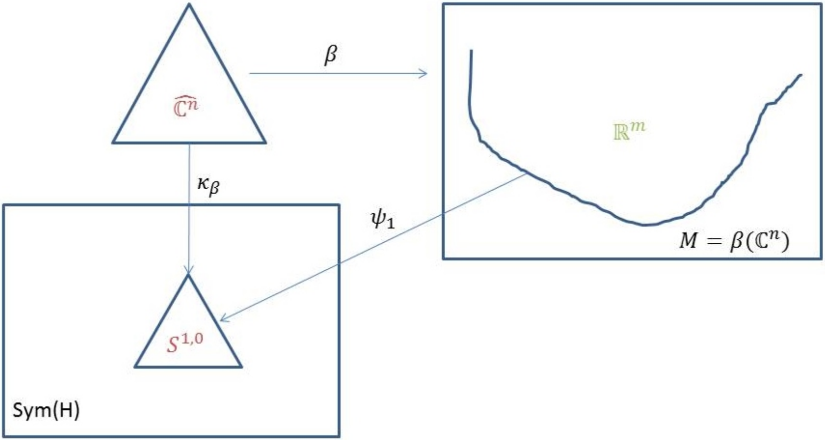
First identify $\hat{H}$ with $\mathcal{S}^{1,0}(H)$.



Proofs

Part 2b: Extension of the inverse for β

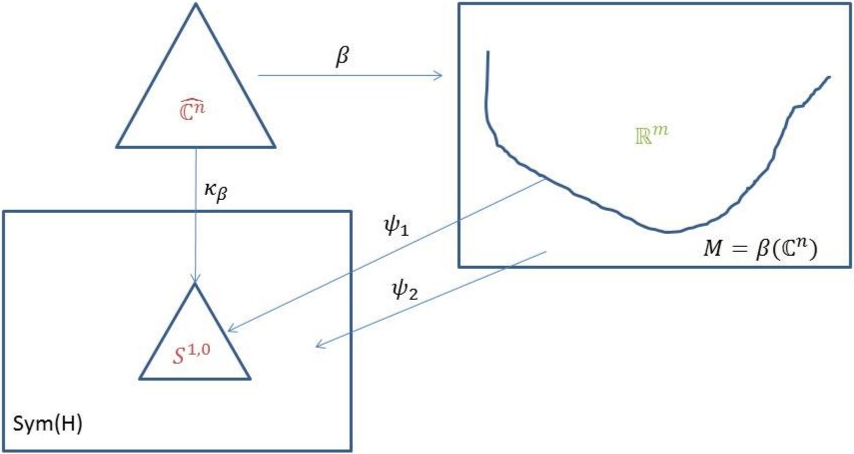
Then construct the local left inverse $\psi_1 : M \to \hat{H}$ with $Lip(\psi_1) = \frac{1}{\sqrt{a_0}}$.



Proofs

Part 2b: Extension of the inverse for β

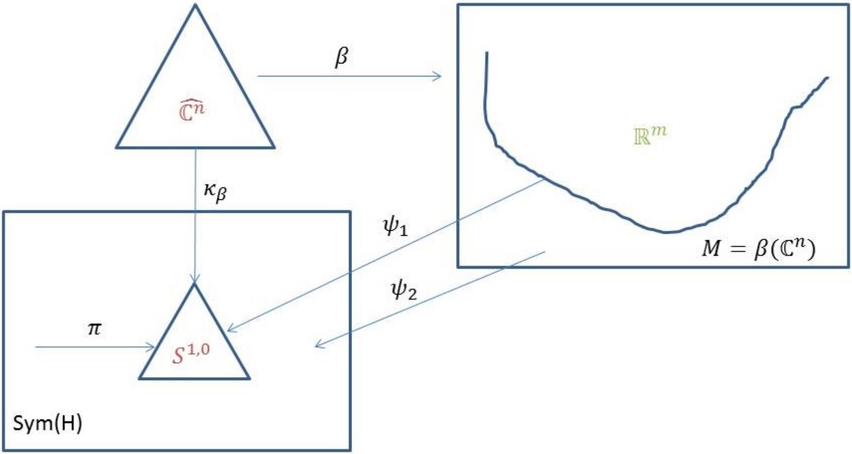
Use Kirszbraun's theorem to extend isometrically $\psi_2 : \mathbb{R}^m \to Sym(H)$.



Proofs

Part 2b: Extension of the inverse for β

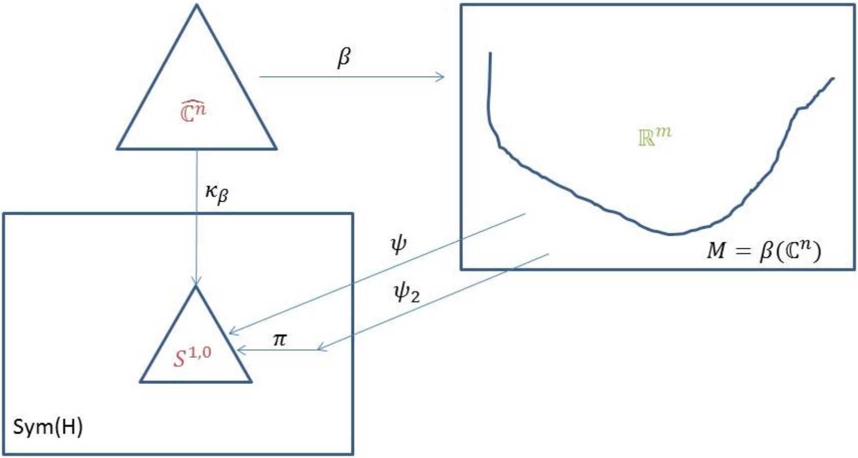
Construct a Lipschitz "projection" $\pi : Sym(H) \to \mathcal{S}^{1,0}(H)$.



Proofs

Part 2b: Extension of the inverse for β

Compose the two maps to get $\psi : \mathbb{R}^m \to \mathcal{S}^{1,0}$, $\psi = \pi \circ \psi_2$.



Part 2: $\mathcal{S}^{1,0}(H)$ as Lipschitz retract in $Sym(H)$

Lemma

Consider the spectral decomposition of the self-adjoint operator A in $Sym(H)$, $A = \sum_{k=1}^{d} \lambda_{m(k)} P_k$. Then the map

$$\pi : Sym(H) \to \mathcal{S}^{1,0}(H) \quad , \quad \pi(A) = (\lambda_1 - \lambda_2) P_1$$

satisfies the following two properties:

1. *for $1 \leq p \leq \infty$, it is Lipschitz continuous from $(Sym(H), d_p)$ to $(\mathcal{S}^{1,0}(H), d_p)$ with Lipschitz constant less than or equal to $3 + 2^{1+\frac{1}{p}}$;*
2. *$\pi(A) = A$ for all $A \in \mathcal{S}^{1,0}(H)$.*

Proof uses Weyl's inequality and spectral formula on a complex integration contour by Zwald & Blanchard (2006).
Recently [2018]: Wenbo Li [AMSC/UMD] proved $Lip(\pi) = 2$ for $p = \infty$.

$\mathcal{S}^{r,0}(H)$ as Lipschitz retract in $Sym(H)$

Lemma

Consider the nonlinear projector P_+ onto the cone of PSD matrices $Sym^+(H)$. Then the map

$$\pi_r : Sym(H) \to \mathcal{S}^{1,0}(H) \quad , \quad \pi(A) = P_+(A - \lambda_{r+1}(A)I)$$

satisfies the following two properties:

1. *for $1 \leq p \leq \infty$, it is Lipschitz continuous from $(Sym(H), \|\cdot\|_p)$ to $(\mathcal{S}^{r,0}(H), \|\cdot\|_p)$;*
2. *$\pi_r(A) = A$ for all $A \in \mathcal{S}^{r,0}(H)$.*

THANK YOU!!

Tensor Products

Consider $A \in \mathbb{C}^{n \times n}$. We seek "optimal" decompositions of A into a sum of rank-1 operators: $A = \sum_k u_k v_k^*$.
Assume A to be positive semi-definite: $A = A^* \geq 0$ ("covariance").
Consider the following three optimization problems:

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Consider the following three optimization problems:
Criterion 1:

$$J(A) = \inf_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2.$$

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Consider the following three optimization problems:
Criterion 1:

$$J(A) = \inf_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2.$$

Criterion 2:

$$J_0(A) = \inf_{A=\sum_{k=1}^m \epsilon_k f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2$$

where $\epsilon_k \in \{+1,-1\}$.

Tensor Products

Consider $A \in \mathbb{C}^{n \times n}$. We seek "optimal" decompositions of A into a sum of rank-1 operators: $A = \sum_k u_k v_k^*$.
Assume A to be positive semi-definite: $A = A^* \geq 0$ ("covariance").
Consider the following three optimization problems:
Criterion 1:

$$J(A) = \inf_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2.$$

Criterion 2:

$$J_0(A) = \inf_{A=\sum_{k=1}^m \epsilon_k f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2$$

where $\epsilon_k \in \{+1, -1\}$.
Criterion 3:

$$J_\wedge(A) = \inf_{A=\sum_{k=1}^m f_k g_k^*} \sum_{k=1}^m \|f_k\|_1 \|g_k\|_1$$

What we know

$$J_\wedge(A) = \min_{A=\sum_{k=1}^m f_k g_k^*} \sum_{k=1}^m \|f_k\|_1 \|g_k\|_1$$

$$J_0(A) = \min_{A=\sum_{k=1}^m \epsilon_k f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2$$

$$J(A) = \min_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2.$$

What we know

$$J_\wedge(A) = \min_{A=\sum_{k=1}^m f_k g_k^*} \sum_{k=1}^m \|f_k\|_1 \|g_k\|_1$$

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$$J(A) = \min_{A=\sum_{k=1}^m f_k f_k^*} \sum_{k=1}^m \|f_k\|_1^2.$$

For every $A \in Sym^+(\mathbb{C}^n)$,

$$\sum_{i,j} |A_{i,j}| =: \|A\|_\wedge = J_\wedge(A) \leq J_0(A) \leq J(A) \leq n\|A\|_\wedge$$

An Open Problem

A remaining open problem:Is there a universal constant $C_0 > 1$ so that for any $n \geq 1$ and every positive semidefinite $A \in \mathbb{C}^{n\times n}$,

$$J(A) = \min_{A=\sum_{k=1}^{m} f_k f_k^*} \|f_k\|_1^2 \leq C_0 \sum_{i,j=1}^{n} |A_{i,j}| \quad ?$$

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Why we care?

If the answer is positive, it follows that, given a trace-class positive semidefinite operator $T : f \mapsto Tf(x) = \int K(x,y)f(y)dy$ the following two statements are equivalent:

1. $K \in M^1(\mathbb{R}^2)$.
2. There are functions $g_k \in M^1(\mathbb{R})$ so that
$$T = \sum_{k\geq 0} \langle \cdot, g_k\rangle g_k$$
and $\sum_{k\geq 0} \|g_k\|_{M^1}^2 < \infty$.

Source Separation Problem: Finding a linear mixing model with minimal "blinding: spots.

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