Assignment #1, due Monday, March 2

1.

(a) Let $F\colon \Omega \to \mathbb{R}^n$ with $\Omega \subset \mathbb{R}^n$. Show that $\left\| D^2F(x)\langle u,v\rangle \right\|_\infty \le C \left\| u\right\|_\infty \left\| v\right\|_\infty$ with

$$C = \max_i \sum_{j,k} \left| \frac{\partial^2 F_i(x)}{\partial x_j \partial x_k} \right|$$

(b) We want to solve the nonlinear system $x_1 = \cos(x_1+x_2)/3$, $x_2 = \sin(x_1-x_2)/3$ using the Newton method with initial guess $x^0 = (0,0)$. Show that you can apply the Newton-Kantorovich theorem and (a). What does the theorem say about the location of x_*?

(c) Perform the Newton iteration from (b) in Matlab with 310 digits accuracy (see example code on the course web page). How many iterations do you need to obtain $\left\| x^k - x_* \right\|_\infty / \left\| x_* \right\|_\infty \le 10^{-300}$? How can you see the convergence order in your computations?

(d) Repeat (c) for the Broyden iteration with $x^0 = (0,0)$, $B_0 = I$ (identity matrix). How many iterations do you need to obtain $\left\| x^k - x_* \right\|_\infty / \left\| x_* \right\|_\infty \le 10^{-300}$? What convergence order do you observe numerically? Does B_k converge to $F'(x_*)$? Check the convergence order of $\left\| (B_k - F'(x_*))\, s_k \right\| / \left\| s_k \right\|$.

2.

(a) Prove the following theorem: Assume that F satisfies assumptions (A1)–(A3) in Ω. Assume that $\gamma \left\| F'(x_*)^{-1} \right\| \left\| x_0 - x_* \right\| < \frac{2}{3}$. Show that the Newton iterates x_k converge q-quadratically to x_*. *Hint:* Use $\gamma \left\| F'(x_*)^{-1} \right\| \left\| x_0 - x_* \right\| \le q$ and prove a modified form of (L2), then use this in the proof of quadratic convergence from class.

(b) Consider the example $F(x) = \arctan x$ for $x \in \mathbb{R}$ with $x_* = 0$. What condition $|x_0 - x_*| < c$ does (a) give for convergence?

3. Let $h_0 \in (0, \frac{1}{2}]$ and $f(t) := \frac{1}{2}t^2 - t + h_0$. Let $\rho_- := 1 - \sqrt{1-2h_0}$.

(a) Consider the Newton iterates t_k for the equation $f(t) = 0$ with initial guess $t_0 = 0$. Show that t_k satisfies $t_{k-1} < t_k < \rho_-$. Show that this implies $\lim_{k\to\infty} t_k = \rho_-$.

(b) Show that t_k converges q-linearly for $h_0 = \frac{1}{2}$. Show that t_k converges q-quadratically for $h_0 < \frac{1}{2}$.

(c) Let $a_0 := 1$, $a_{k+1} := a_k(1-h_k)$, $h_{k+1} := \frac{1}{2}h_k^2/(1-h_k)^2$. Show that $t_{k+1} - t_k = h_k a_k$ for $k = 0, 1, 2, \ldots$.

Hint: Let $d_k := a_k h_k$, $\tau_0 := 0$, $\tau_{k+1} := \tau_k + d_k$. First show that $a_k - a_{k+1} = d_k$ and hence $\tau_k = 1 - a_k$. Then show that $d_k = \frac{1}{2}d_{k-1}^2/(1-\tau_k)$. On the other hand show from the definition of t_k that $t_{k+1} = \frac{\frac{1}{2}t_k^2 - h_0}{t_k - 1}$ which implies $\frac{1}{2}(t_{k+1}-t_k)^2 = \frac{1}{2}t_{k+1}^2 - t_{k+1} + h_0$ and hence $t_{k+1} - t_k = \frac{1}{2}(t_k - t_{k-1})^2/(1-t_k)$.

(d) Complete the proof of the Newton-Kantorovich theorem from class: Let $a_k = \omega_k^{-1}$, w.l.o.g. $\omega_0 = 1$. We showed that $\left\| x_{k+1} - x_k \right\| \le d_k$ for $k = 0, 1, 2, \ldots$ with d_k as defined in (c). Show that $\left\| x_0 - x_k \right\| < \rho_-$ for $k = 0, 1, \ldots$. Show that x_k form a Cauchy sequence which converges to x_* with $\left\| x_0 - x_* \right\| \le \rho_-$. Show that the convergence is r-linear for $h_0 = \frac{1}{2}$, and r-quadratic for $h_0 < \frac{1}{2}$.