Worksheet 11.3 – 11.4 Solutions

0 $I = \int_0^{\pi/4} \frac{\tan x}{\sqrt{\sec x}}\,dx$ let $u = \sec x$ then $du = \sec x \tan x\,dx$

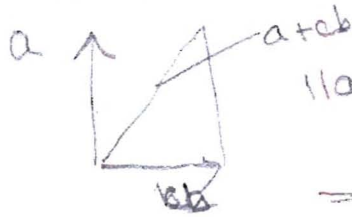
$\tan x\,dx = \frac{du}{u}$. $x = 0$ $u = 1$, $x = \pi/4$, $u = \sqrt{2}$

$$I = \int_1^{\sqrt{2}} \frac{du}{u^{3/2}} = -\frac{2}{u^{1/2}}\Big|_1^{\sqrt{2}} = 2 - 2^{3/4}$$

1) i let $u = i$ $v = j$. $w = j$ $(u \times v) \times w = (i \times j) \times j = k \times j = -i$
while $u \times (v \times w) = i \times (j \times j) = i \times 0 = \underline{0}$

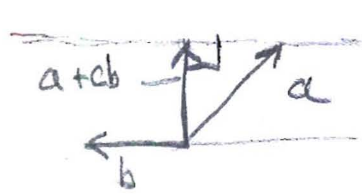
1a) Scalar b) Scalar c) vector d) undefined
e) vector f) undefined g) vector h) undefined
i) undefined j. undefined

2 a) $\|a + cb\|^2 = (a + cb)\cdot(a + cb) = \|a\|^2 + 2c\, a\cdot b + c^2 \|b\|^2$
if $a\cdot b = 0$ $\|a + cb\|^2 = \|a\|^2 + c^2\|b\|^2 \geq \|a\|^2$, if $a \cdot b \neq 0$
let $f(c) = \|a\|^2 + 2c\cdot a\cdot b + c^2\|b\|^2$
$f'(c) = 2a\cdot b + 2c\|b\|^2 = 0$ $c = -\frac{a\cdot b}{\|b\|^2}$ for min
$f\left(\frac{a\cdot b}{\|b\|^2}\right) = \|a\|^2 - 2\frac{(a\cdot b)^2}{\|b\|^2} + \frac{(a\cdot b)^2}{\|b\|^2} = \|a\|^2 - \frac{(a\cdot b)^2}{\|b\|^2} < \|a\|^2$ (or see c Below)



$\|a + cb\|^2 = \|a\|^2 + c^2\|b\|^2 \geq \|a\|^2$

a + cb



c) choose c so that $a + cb \perp b$
$(a + cb)\cdot b = 0 \Rightarrow c = \frac{-a\cdot b}{\|b\|^2}$

a+cb is a Side of a Right △ a is the hypotenuse.

3 $u = 2i - j + k$, $v = i + j + 2k$ $w = 2i - j + 4k$.

$u\cdot u = 6$ $v\cdot v = 6$
$u\cdot v = 3$ $u\cdot w = 9$ $v\cdot w = 9$

$(w - au - bv)\cdot u = 0$ $a\, u\cdot u + b\, u\cdot v = u\cdot w$
$(w - au - bv)\cdot v = 0$ $a\, u\cdot v + b\, v\cdot v = v\cdot w$

$6a + 3b = 9$ $2a + b = 3$ $\Rightarrow$ $\underline{\underline{a = b = 1}}$
$3a + 6b = 9$ $a + 2b = 3$

($w - (u + v) = -i - j + k$
$\perp$ to both u and v)

Worksheet 11.3 / 11.4

(2a) Proof: Assume two nonzero vectors $\vec{a}$ and $\vec{b}$ are perpendicular. Let $c \in \mathbb{R}$.

Then $\vec{a}\cdot\vec{b} = 0$, in particular, $2c\,\vec{a}\cdot\vec{b} = 0$.

Now, $0 \le c^2\|\vec{b}\|^2$ for any c, $\vec{b}$.

$$\Rightarrow \quad 0 \le 2c\,\vec{a}\cdot\vec{b} + c^2\|\vec{b}\|^2$$

$$\begin{aligned} \|\vec{a}\|^2 &\le \|\vec{a}\|^2 + 2c\,\vec{a}\cdot\vec{b} + c^2\|\vec{b}\|^2 \\ &= \vec{a}\cdot\vec{a} + 2c\,\vec{a}\cdot\vec{b} + c^2\,\vec{b}\cdot\vec{b} \\ &= (\vec{a}+c\vec{b})\cdot(\vec{a}+c\vec{b}) \\ &= \|\vec{a}+c\vec{b}\|^2 \end{aligned}$$

$$\Rightarrow \quad \|a\| \le \|\vec{a}+c\vec{b}\|$$

To prove the converse, assume $\|\vec{a}\| \le \|\vec{a}+c\vec{b}\|$ for any two nonzero vectors $\vec{a}$ and $\vec{b}$ and any $c \in \mathbb{R}$.

Then
$$\begin{aligned} \|\vec{a}\|^2 &\le \|\vec{a}+c\vec{b}\|^2 \qquad \boxed{1} \\ &= (\vec{a}+c\vec{b})\cdot(\vec{a}+c\vec{b}) \\ &= \vec{a}\cdot\vec{a} + 2c\,\vec{a}\cdot\vec{b} + c^2\vec{b}\cdot\vec{b} \\ &= \underbrace{\|\vec{a}\|^2 + 2c\,\vec{a}\cdot\vec{b} + c^2\|\vec{b}\|^2}_{\equiv f(c)} \end{aligned}$$

$$\Rightarrow \quad f'(c) = 2\vec{a}\cdot\vec{b} + 2c\|\vec{b}\|^2$$

Setting $f'(c_0) = 0 \Rightarrow c_0 = \dfrac{-\vec{a}\cdot\vec{b}}{\|b\|^2}$ is a crit. pt.

$$\Rightarrow f(c_0) = \|\vec{a}\|^2 + 2\left(\frac{-\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{a}\cdot\vec{b} + \left(\frac{-\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\right)^2\|\vec{b}\|^2$$

$$= \|\vec{a}\|^2 - \frac{2(\vec{a}\cdot\vec{b})^2}{\|\vec{b}\|^2} + \frac{(\vec{a}\cdot\vec{b})^2\,\cancel{\|\vec{b}\|^2}}{\|\vec{b}\|^{\cancel{4}\,2}}$$

$$= \|\vec{a}\|^2 - \frac{(\vec{a}\cdot\vec{b})^2}{\|\vec{b}\|^2}$$

(<u>Note:</u> $f(c_0)$ is the minimum of $\|\vec{a}+c\vec{b}\|^2$.)

From $\boxed{1}$ it follows that

$$\|\vec{a}\|^2 \le \|\vec{a}\|^2 - \frac{(\vec{a}\cdot\vec{b})^2}{\|\vec{b}\|^2}$$

$$0 \le -\frac{(\vec{a}\cdot\vec{b})^2}{\|\vec{b}\|^2}$$

$$\frac{(\vec{a}\cdot\vec{b})^2}{\|\vec{b}\|^2} \le 0$$

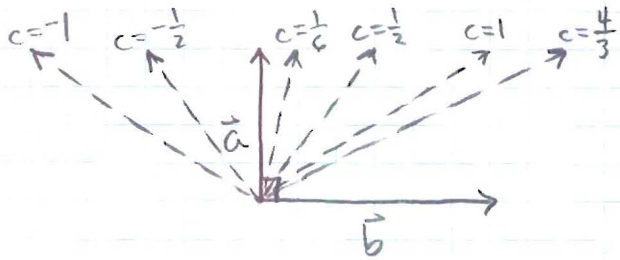
$$(\vec{a}\cdot\vec{b})^2 \le 0$$

But $(\vec{a}\cdot\vec{b})^2$ is nonnegative, so $\vec{a}\cdot\vec{b}$ must be 0.

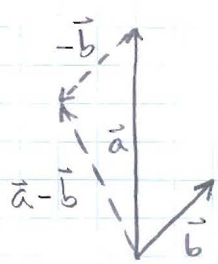
Therefore, $\vec{a}$ and $\vec{b}$ are perpendicular.

$\blacksquare$

2b



2c



$\vec{a}\cdot\vec{b} \neq 0$

Let $c = -1$

$\|\vec{a}\| \not\le \|\vec{a}-\vec{b}\|$

③ A vector orthogonal to both $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -3\hat{\imath} - 3\hat{\jmath} + 3\hat{k}$$

Set $\vec{w} - a\vec{u} - b\vec{v} = c(\vec{u} \times \vec{v})$ for some $c \in \mathbb{R}$, $c \neq 0$.

$$\Rightarrow \begin{aligned} 2 - 2a - b &= -3c \quad \boxed{1} \\ -1 + a - b &= -3c \quad \boxed{2} \\ 4 - a - 2b &= 3c \quad \boxed{3} \end{aligned}$$

$\boxed{2} + \boxed{3} \Rightarrow 3 - 3b = 0 \Rightarrow b = 1$

$\boxed{1} - \boxed{2} \Rightarrow 3 - 3a = 0 \Rightarrow a = 1$