# Worksheet for Sections 15.1-15.2

KEY

1. Let $\mathbf{F} = y\sin z\mathbf{i} + (x\sin z + z)\mathbf{j} + (xy\cos z + y + 2z)\mathbf{k}$. Determine whether $\mathbf{F}$ is the gradient of some function $f$. If it is, find $f$.

3

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y\sin z & (x\sin z + z) & (xy\cos z + y + 2z) \end{vmatrix} \checkmark$$

$$= \hat{i}(x\cos z + 1 - x\cos z - 1) - \hat{j}(y\cos z - y\cos z) + \hat{k}(\sin z - \sin z) = \vec{0}$$

$$\frac{\partial f}{\partial x} = y\sin z$$

$$f = xy\sin z + g(y,z) = xy\sin z + yz + h(z)$$

$$\frac{\partial f}{\partial y} = x\sin z + \frac{\partial g}{\partial y} = x\sin z + z$$

$$g = yz + h(z)$$

$$\frac{\partial f}{\partial z} = xy\cos z + y + h'(z) = xy\cos z + y + 2z$$

$$h(z) = z^2 + C$$

$$\boxed{f(x,y,z) = xy\sin z + yz + z^2 + C} \checkmark\checkmark$$

2. Let $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$. Show that

$$\Delta f = \nabla\cdot\nabla f = 0$$

4

Note: $\Delta f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$

$$\partial_x f = -\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2x$$

$$\partial_x^2 f = -(x^2+y^2+z^2)^{-3/2} + \tfrac{3}{2}(x^2+y^2+z^2)^{-5/2}\, 2x^2 \checkmark\checkmark$$

$$\Rightarrow \Delta f = -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-5/2}[x^2+y^2+z^2] \checkmark\checkmark$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-3/2} = 0 \checkmark$$

3.Let a force $\mathbf{F}$ be given by

$$\mathbf{F}(x,y,z) = (-)y\mathbf{i} + x\mathbf{j} + z^3\mathbf{k}.$$

Find the work $W$ done by the force $\mathbf{F}$ on an object that moves from $(1,0,0)$ ($P_0$) to $(0,1,\pi)$ ($P_1$)

2 (a) along a straight line.

2 (b) along a helix parameterized by $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$ for $0 \le t \le \pi/2$.

$\vec{P_0P_1} = (-1, 1, \pi)$

(a) $W = \int_a^b \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt$

$\vec{r}(t) = \vec{P_0} + t\vec{P_0P_1}, \quad 0 \le t \le 1$

$= (1-t)\hat{i} + t\hat{j} + \pi t\hat{k}$

$\frac{d\vec{r}}{dt}(t) = -\hat{i} + \hat{j} + \pi\hat{k}$

$$= \int_0^1 (-t, 1-t, \pi^3 t^3)\cdot(-1,1,\pi)\,dt \checkmark$$

$$= \int_0^1 (t + 1 - t + \pi^4 t^3)\,dt$$

$$= \left[t + \frac{\pi^4}{4}t^4\right]_0^1 = \boxed{1 + \frac{\pi^4}{4}} \checkmark$$

(b) $\frac{d\vec{r}}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 2\hat{k}$

$$W = \int_0^{\pi/2} (-\sin t, \cos t, 8t^3)\cdot(-\sin t, \cos t, 2)\,dt \checkmark$$

$$= \int_0^{\pi/2} (\sin^2 t + \cos^2 t + 16t^3)\,dt$$

$$= \int_0^{\pi/2} (1 + 16t^3)\,dt$$

$$= \left[t + 4t^4\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \frac{4\pi^4}{16} = \boxed{\frac{\pi}{2} + \frac{\pi^4}{4}} \checkmark$$