Research Statement

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My current research is the study of weight varieties, which are quotients of flag manifolds G/P by a maximal torus T, where G is a complex reductive Lie group and P is a parabolic subgroup of G containing T. I have mainly studied the algebraic geometry of weight varieties; in particular I have analyzed properties of their projective coordinate rings.

I have completed three papers. The most recent is *"Matroids and Geometric Invariant Theory of torus actions on flag spaces"*, available as a preprint at http:///xxx.lanl.gov/abs/math.AG/0511608. The results in this paper will be used in my doctoral thesis. Here I study the structure of a G.I.T. quotient of a flag manifold $F = \mathrm{SL}(n, \mathbb{C})/P$ by the natural action of the maximal torus T in $\mathrm{SL}(n, \mathbb{C})$. The construction of the quotient space depends upon the choice of a T-linearized line bundle L of F. I investigate the case where L is a very ample homogeneous line bundle. A theorem of Gel'fand, Goresky, MacPherson, and Serganova about matroids and their respective polytopes is applied to study semistability of flags relative to a given T-linearization of L. The main theorem of this note is that regardless of the choice of linearization, the semistable flags are detected by invariant sections of L; that is, for each semistable flag $p \in F$, there is a T-invariant section s of L such that $s(p) \neq 0$. In particular the degree one elements of the projective coordinate ring determine a well-defined map to projective space. Additionally I find that the closure of any T-orbit in F is projectively normal for any projective embedding of F.

The second most recent, *"The projective invariants of ordered points on the line"*, http://xxx.lanl.gov/abs/math.AG/0505096, is co-authored by John J. Millson, Andrew Snowden, and Ravi Vakil. In this paper we study the relations among the generators (discovered to be those of lowest degree by A. Kempe in 1894) for the ring of projective invariants of n ordered points on the projective line. We show that if the number of points is odd the relations are quadratic and if the number of points is even the relations have degree at most four. We outline an algorithm for listing the relations. We show that the above ring is a complete intersection if and only if n=1,2,3,4,6. The moduli space of n ordered points on the Riemann sphere modulo $\mathrm{PGL}(2,\mathbb{C})$ is a weight variety since it is isomorphic to a torus quotient of the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$ via the Gel'fand–MacPherson correspondence.

We are currently revising a related paper, "The space of ordered points on the line is almost always cut out by quadrics", where we prove that linear relations and certain quadric binomial relations cut out the projective scheme of the moduli space except for the case of six points, where there is a necessary cubic relation. We conjecture that these quadric binomials actually generate the ideal. This paper is available on my homepage, http://www.math.umd.edu/∼bhoward, in pdf format.

The third completed paper is co-authored with John J. Millson, entitled *"The Chevalley involution and a duality of weight varieties"*. It was published in the Asian Journal of Math, Armand Borel memorial issue, Vol. 8 No. 4, Dec 2004. In this paper we show that the classical notion of association of projective point sets is a special case of a general duality between weight varieties. There are three types of isomorphism theorems, the first type is a Kähler isomorphism of symplectic quotients, the second type is an algebraic isomorphism of Mumford quotients and the third type is an explicit formula for the isomorphism of homogeneous coordinate rings in terms of combinatorial Lie theory. In those cases where there is a self duality (the weight variety maps to itself) we classify when the duality map is the identity map. In particular we show that all self dualities are nontrivial for the weight varieties associated to the exceptional groups.

I have nearly completed a fourth paper, *"The toric geometry of polygons in Euclidean space"*, with John J. Millson and Christopher Manon. We study the topology of the toric fibers of certain toric degenerations of moduli spaces of Euclidean polygons with prescribed side lengths. We prove a conjecture of Phillip Foth and Yi Hu that these toric varieties are a certain topological quotient (constructed by Kamiyama and Yoshida) of their associated polygon moduli spaces. The quotient construction is called symplectic implosion, due to Guillemin, Jeffrey, and Sjamaar.

Future Research:

The paper *"The projective invariants of ordered points on the line"* has at least two possible generalizations. One is to moduli spaces attached to curves, the other is to weight varieties. The invariant theory problem for weight varieties (finding finite sets of generators and relations for the subring of T–invariants) appears to be too difficult to solve in general. In particular, the lowest degree invariants do not generate the subring of invariants except in very special cases. However, I hope to solve the associated problem for n ordered points on $\mathbb{CP}^k$ for $k \geq 2$, a fundamental problem in classical algebraic geometry.

There is the (extremely difficult) problem of finding a presentation for the ring of n *un-ordered* points on the projective line, which may be viewed as the space of homogeneous degree n polynomials in two variables modulo $\mathrm{SL}(2,\mathbb{C})$, by expressing invariants in terms of roots. One approach is to take the ring of ordered points and restrict to the invariants under the action of the symmetry group on n letters. This is the classical problem of binary quantics. Various mathematicians have solved this problem for $n = 5, 6, 8$. Perhaps $n = 10$ is now within reach.

John J. Millson and I intend to study the graded coordinate ring of the moduli space of parabolically semistable rank two degree zero vector bundles over $\mathbb{CP}^1$ with n marked points (there are connections with the subject of conformal field theory). We believe that the method we employed in *"The projective invariants of ordered points on the line"* (which was to degenerate to a toric variety and solve the presentation problem for the associated toric ring) will generalize to this case.

Also, the result from the paper *"Matroids and Geometric Invariant Theory of torus actions on flag spaces"* could perhaps be generalized to groups other than $\mathrm{SL}(n,\mathbb{C})$. First of all it would be nice to know exactly which tensor power of the line bundle detects the semistable points. For example, John J. Millson and I have shown that the second tensor power is sufficient for $G = \mathrm{Sp}(2n,\mathbb{C})$ a symplectic group, and the fourth tensor power is sufficient for $G = \mathrm{SO}(n,\mathbb{C})$ an orthogonal group. We would like to know the answer for the exceptional groups as well. (We expect to write a paper containing these result in the near future.) These results may be connected to the famous saturation problems of representation theory, where one would like to decide if a given irreducible representation occurs as a summand in the tensor product of two other irreducible representations. Additionally, it is an open problem whether or not every T-orbit closure in G/P is projectively normal for G not $\mathrm{SL}(n,\mathbb{C})$.

Sincerely,

Benjamin J. Howard