AMSC 466. Spring 2020

Homework Assignment 6. Due Thursday April 2.

1. Let A be a real symmetric $n \times n$ matrix. A powerful algorithm for finding a specific eigenpair of A starting from an initial approximation for the eigenvector is the Rayleigh quotient iteration. The Rayleigh quotient is a function $Q : \mathbb{R}^n \to \mathbb{R}$ defined by

$$Q(x) := \frac{x^\top A x}{x^\top x}. \tag{1}$$

The Rayleigh quotient iteration algorithm is written below. The norm $\|\cdot\|$ in it is the 2-norm.

Input *An initial guess for the desired eigenvector:* $w \in \mathbb{R}^n$.
A symmetric $n \times n$ *matrix* A.
Tolerance `tol`.
Initialization *Normalize the approximation to eigenvector:* $v = w/\|w\|$;
Compute approximation to eigenvalue: $\lambda = v^\top A v$;
Compute the residual: $r = Av - \lambda v$;
The main body
while $\|r\| >$ `tol` **do**
1: Solve $(A - \lambda I)w = v$ for w;
2: Normalize: $v = w/\|w\|$;
3: Update λ: $\lambda = v^\top A v$;
4: Recompute the residual: $r = Av - \lambda v$;
end

Algorithm 1: Rayleigh quotient iteration

Remark The Rayleigh quotient iteration converges cubically, i.e., the order of convergence is 3. This is a rare luck!

(a) **(2 pts)** Show that if $\{\lambda, v\}$ is an eigenpair of A then $\{(\lambda - \mu)^{-1}, v\}$ is an eigenpair of $(A - \mu I)^{-1}$.

(b) **(3 pts)** Verify that if v is an eigenvector of A then: (i) $Q(v) = \lambda$, the corresponding eigenvalue, and (ii) $\nabla Q(v) = 0$, i.e., v is a stationary point of $Q(x)$.

(c) **(5 pts)** In step 1 of the Rayleigh quotient algorithm, we solve the system $(A - \lambda I)w = v$. The matrix $(A - \lambda I)$ is close to singular if λ is close to an actual eigenvalue of A. However, step (1) is not ill-conditioned. Explain why using your knowledge of condition numbers.

(d) **(5 pts)** Let A be a $N \times N$ matrix of the form

$$A := \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{bmatrix}.$$

This matrix often arises in finite difference methods: $h^{-2}A$ ($h = 1/N$ is the step in space) is the approximation to the second derivative operator.

The eigenvalues and eigenvectors of A are well-known: for $k = 1, 2, \ldots, N$,

$$\lambda_k = -4\left(\sin\frac{\pi k}{2N+2}\right)^2, \quad v_k = \left[\sin\left(\frac{\pi k}{N+1}\right), \sin\left(\frac{2\pi k}{N+1}\right), \ldots, \sin\left(\frac{N\pi k}{N+1}\right)\right]^\top. \tag{2}$$

Set $N = 100$. This matrix A is set up in Matlab using the commands (see Matlab's help on `spdiags` if you are not familiar with this command):

```
N = 100;
e = ones(N,1);
A = spdiags([e,-2*e,e],-1:1,N,N);
```

Pick the initial approximation for an eigenvector:

```
x = linspace(0,1,N)';
v = x.*(1-x);  % '.*' means that the multiplication is performed entrywise
```

Implement the Rayleigh quotient algorithm to compute an eigenpair starting from this approximation. Display the found eigenvalue with 15 digits after comma (format `%.15e`). To which eigenvalue of A did the algorithm converge? How many iterations were performed?

Do the same task for $v = x(1-x)(x-1/2)$ and $v = x(1-x)(x-1/3)(x-2/3)$.

Submit your code and its printout.

2. **(10 pts)** Problem 1 in Chapter 5 in `BindelGoodman.pdf` (page 123) posted on ELMS. *Hint: to do (a), count flops in reducing the* $n \times (2n)$ *matrix* $[A, I]$ *to* $[U, L^{-1}]$ *using Gaussian elimination.*