Take-home Final exam. Problem 1. Due May 17, 11:59 PM

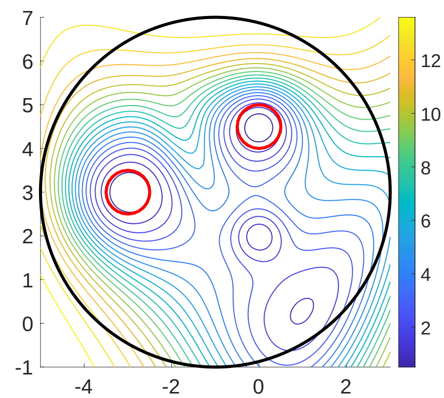
Consider the following Boundary Value Problem (BVP) in 2D:

$$\Delta u - \nabla V(x,y)\cdot\nabla u = 0, \quad (x,y)\in\Omega \tag{1}$$

$$\frac{\partial u}{\partial n} = 0, \quad u \in \Gamma_N \tag{2}$$

$$u(\partial A) = 0, \quad u(\partial B) = 1, \tag{3}$$

The domain Ω is bounded by the black circle and the two red circles shown in the figure. The boundary Γ_N is the black circle, while the boundaries ∂A and ∂B are the left and the right red circles, respectively. The level sets of the function $V(x,y)$, the "face potential", are color-coded.



$$\partial A: \ \text{center}: \ (-3,3), \ \text{radius}: \ 0.5;$$
$$\partial B: \ \text{center}: \ (0,4.5), \ \text{radius}: \ 0.5;$$
$$\Gamma_N: \ \text{center}: \ (-1,3), \ \text{radius}: \ 4.$$

The function $V(x,y)$ is given by:

```
xa=-3; ya=3;
xb=0; yb=4.5;
fac=10;
f=(1-x).^2+(y-0.25*x.^2).^2+1;
g1=1-exp(-0.125*((x-xa).^2+(y-ya).^2));
g2=1-exp(-0.25*(((x-xb).^2+(y-yb).^2)));
g3=1.2-exp(-2.*((x+0).^2+(y-2).^2));
g4=1+exp(-2*(x+1.5).^2-(y-3.5).^2-(x+1).*(y-3.5));
v1=f.*g1.*g2.*g3.*g4;
V=fac*atan(v1/fac);
```

Remark. The solution to this problem has the following probabilistic interpretation. Suppose a particle is moving inside the region Ω with reflecting boundary Γ_N according to the overdamped Langevin dynamics $\dot{z} = -\nabla V(z) + \sqrt{2}\eta$ where η is two-dimensional white noise. The potential force $-\nabla V(z)$ pushes it downhill while the white noise pushes it randomly in different directions. Suppose at time $t = 0$ the particle is at the point $z := (x,y)$. Then $u(x,y)$ is the probability that the particle will reach the region B before reaching region A. The function $u(x,y)$ has a special name: the *committor* function.

1. **(5 pts)** Show that Eq. (1) is equivalent to

$$\nabla \cdot \left(e^{-V(x,y)} \nabla u \right) = 0. \tag{4}$$

2. **(5 pts)** Let u be the solution of Eq. (4) with BCs (2)-(3) and u_D be a smooth function such that $u_D = 1$ at ∂B and $u_D = 0$ outside a small neighborhood of ∂B. Define $v := u - u_D$. Note that v satisfies $v(\partial B) = v(\partial A) = 0$ and $\frac{\partial v}{\partial n} = 0$ on Γ_N. Show that then for any continuous and piecewise continuously differentiable function w defined on Ω such that $w(\partial A) = w(\partial B) = 0$

$$\int_\Omega e^{-V(x,y)} \nabla v \cdot \nabla w dxdy = -\int_\Omega e^{-V(x,y)} \nabla u_D \cdot \nabla w dxdy. \tag{5}$$

3. **(10 pts)**Triangulate the domain Ω and solve the BVP (4),(2),(3) using the finite element method (FEM). Plot the triangulation and the FEM solution.

4. **(10 pts)** To find the FEM solution, you need to solve a system on linear equations $A\mathbf{u} = \mathbf{b}$ where A is the stiffness matrix and $\mathbf{b}$ is the load vector. Program the conjugate gradient algorithm with incomplete Cholesky preconditioner (see the Matlab function `ichol`) and use it to solve $A\mathbf{u} = \mathbf{b}$. Set the stopping criterion when the norm of the residual is less than `tol = 1e-12`. Plot the residuals versus the iteration number. Use the `log` for the y-axis.

Submit a single pdf file with your report containing figures, calculations, specifications of methods used, and links to your codes.

AMSC661, Spring 2022 Maria Cameron

Take-home Final exam. Problem 2. Due May 17, 11:59 PM

1. **(10 pts)** Consider the nonviscous Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad 0 \le x < 4, \quad t \ge 0, \tag{1}$$

with periodic boundary conditions $u(0,t) = u(4,t)$ and the initial condition

$$u(x,0) = u_0(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x \notin [0,1]. \end{cases} \tag{2}$$

Plot the shock diagram, i.e, characteristics and shock lines (curves), in the (x,t)-domain $[0,4] \times [0,12]$. Provide equations for all shock lines and analytical solutions in each region bounded by shock lines.

2. **(10 pts)** Solve this problem numerically and plot the numerical solution and the analytical solution in the same figure at times $t = 7$ and $t = 11$. Include legend.

3. **(10 pts)** Now consider viscous Burgers equation

$$u_t + uu_x = 0.01u_{xx}, \quad 0 \le x < 4, \quad t \ge 0, \tag{3}$$

in the same periodic domain and the same initial and boundary conditions. Solve this problem using the spectral method similar to the one you used for solving the Kuramoto-Sivashinsky equation. Plot the numerical solution at time $t = 7$ and $t = 11$ in the same figure as the plots from the previous task.

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