Ellipticity Condition Implications for the Study of Anisotropic Minimal Surfaces

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Abstract

In the study of regularity theory for anisotropic minimal surfaces, several ellipticity conditions have been posed, including the atomic condition (AC), the scalar atomic condition (SAC), the scalar atomic condition 1 (SAC1), and the uniform scalar atomic condition (USAC). In exploring the implications between these ellipticity conditions, we find counterexamples to several open implications. We show that the condition (SAC1) cannot be weakened further to be implied by (SAC), prove that (AC1) is open in the C^1 topology, and in codimension 1 show that (SAC) and (AC) are not open in the C^2 topology.

Background

- **Plateau's Problem**
  Given a boundary γ, what is the surface with that boundary that minimizes the total area? What if we wish to minimize other quantities given the boundary γ?
- **Varifold**
  A d-dimensional varifold V is a Radon measure on $\mathbb{R}^n \times \mathbb{G}(n,d)$, where $\mathbb{G}(n,d)$ is the Grassmannian of d-planes in $\mathbb{R}^n$.
  By disintegration of measure, V be decomposed as
  $$V(dx, dT) = ||V||(dx) \otimes \mu(dT)$$
- **Integrands and Energy**
  Unless otherwise stated, assume all integrands are autonomous: Given a positive C^1 integrand function $\Psi : \mathbb{R}^n \times \mathbb{G}(n,d) \to \mathbb{R}$, the *anisotropic energy* of a varifold V is defined as
  $$\mathbf{\Psi}(V) := \int_{\mathbb{G}(n,d)} \Psi(T) d\mu(T)$$
  Given autonomous integrand Ψ, its dual $\Psi^* : \mathbb{G}(n,d) \to \mathbb{R}$ is defined as
  $$\Psi^*(P) := \Psi(\mathrm{id} - P) = \Psi(P^\perp) \quad \forall P \in \mathbb{G}(n,d)$$
  The *first variation* of the autonomous energy Ψ at the varifold V is given by
  $$\delta_\Psi V(g) = \frac{d}{dt}\mathbf{\Psi}(\varphi_t^\# V, \Omega)\Big|_{t=0} = \int_{\mathbb{G}(n,d)} \left[\langle B_\Psi(T), Dg(x)\rangle\right] d\mu(T)$$
  where
  $$\langle B_\Psi(T), L\rangle := \Psi(T)\langle T, L\rangle + \langle D\Psi(T), T^\perp LT + (T^\perp LT)^\top\rangle \quad \forall L \in \mathbb{R}^{n\times n}$$
  From this,
  $$A(\mu) = \int B_\Psi(T) dV(T)$$
- **Atomic Conditions**
  Ψ satisfies the *atomic condition* (AC) if:
  - (AC1): $\dim\ker A(\mu) \le n-d$ for all $\mu \in \mathcal{P}(\mathbb{G}(n,d))$,
  - (AC2): if $\dim\ker A(\mu) = n-d$, then $\mu = \delta_{T_0}$ for some $T_0 \in \mathbb{G}(n,d)$.

  Ψ satisfies the *scalar atomic condition* (SAC) if
  $$\langle B_\Psi(T), B_{\Psi^*}(S^\perp)\rangle > 0 \quad \forall T \ne S \in \mathbb{G}(n,d)$$
  Ψ satisfies the *scalar atomic condition 1* (SAC1) if there exists $\delta < \frac{1}{d-1}$ such that
  $$\langle B_\Psi(T)w, w\rangle \le (1+\delta)\Psi(T)||w||^2 \quad \forall T \in \mathbb{G}(n,d), w \in \mathbb{R}^n$$

Ellipticity Relations

The following theorems give some known relations among the conditions:
Theorems (De Rosa, Tione, 2020):

- If Ψ is a positive integrand satisfying (SAC), then Ψ satisfies (AC).
- If Ψ is a positive integrand staisfying (SAC1), then Ψ satisfies (AC1).

Proposition: In codimension 1, (SAC) does not imply (SAC1).

Definition: An integrand Ψ is said to satisfy the *weak (SAC1) condition* (wSAC1) if for any orthonormal set of vectors $\{v_1, \ldots, v_{d-1}\}$ and for any μ,
$$\sum_{i=1}^{d-1}\langle A(\mu)v_i, v_i\rangle \le (d-1)(1+\delta)\int \Psi(S)d\mu(S)$$
where $c > n-d$ and $\delta < \frac{1}{d-1}$.

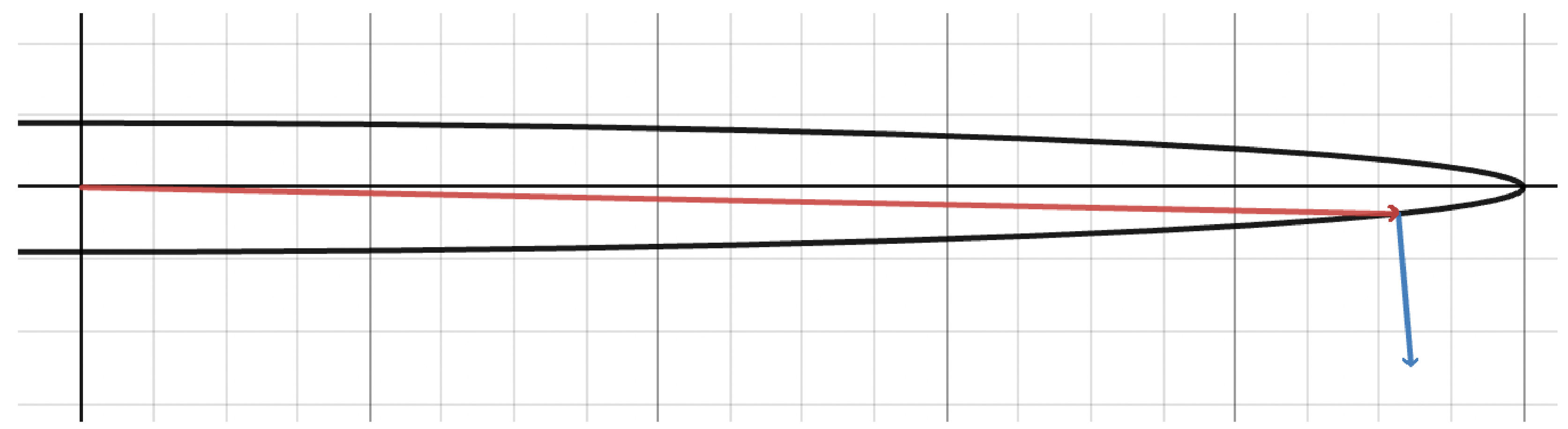

Figure 1. Vector v (in red) and gradient $\nabla_v G(v)$ normal to the surface (in blue). As the ellipse is elongated, the magnitude of the vector and gradient increase while they grow increasingly orthogonal, bounding the inner product on the left hand side of the last expression, while the right hand side can be made arbitrarily large.

In codimension 1, (wSAC1) is equivalent to the following condition holding for $w \in S^{n-1}$, $v \perp u \in S^{n-1}$:
$$(1+(n-2)\delta)G(w) \ge \langle\nabla G(w), v\rangle\langle w, v\rangle + \langle\nabla G(w), u\rangle\langle w, u\rangle$$

Theorem: If Ψ is a positive integrand satisfying (wSAC1) then Ψ satisfies (AC1).

Proposition: In dimension and codimension 1, (SAC) does not imply (wSAC1).

Integrands under Linear Transformations

One question of interest is whether or not ellipticity conditions are invariant under linear transformations.
Lemma: If $\tilde\Psi(v) = \Psi(Av)$, then $B_{\tilde\Psi}(v) = A^\top B_\Psi(Av)(A^{-1})^\top$.

Theorem: If $\Psi \in C^1(\mathbb{G}(n,d))$ satisfies (SAC) and $A \in GL_n(\mathbb{R})$, then $\tilde\Psi(v) = \Psi(Av) \in C^1(\mathbb{G}(n,d))$ satisfies (SAC) as well.

Theorem: Let $A \in O(n)$ and $\Psi \in C^1(\mathbb{G}(n,d))$ satisfy (SAC1). Then there exists $\varepsilon > 0$ such that for all $L_\varepsilon \in \mathbb{R}^{n\times n}$ with $||L_\varepsilon|| < \varepsilon$, $\Psi((A + L_\varepsilon))(v)$ satisfies (SAC1) aswell.

An important consequence:

Proposition: In any codimension, (SAC) does not imply (SAC1) or (wSAC1).

Relations between Conditions in Dimension and Codimension 1

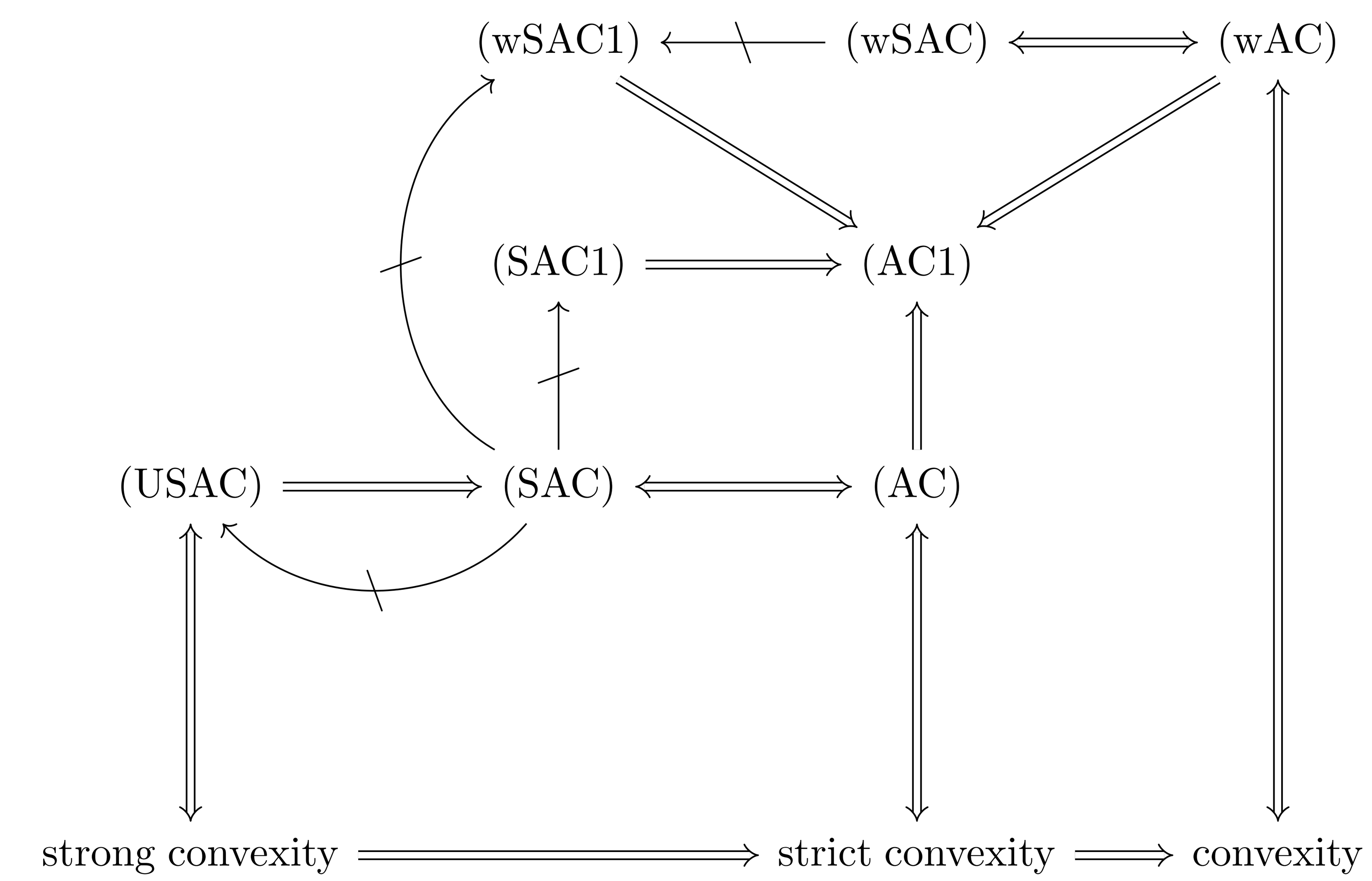

Figure 2. Implication graph.

Topological Properties of Ellipticity Conditions

In general dimension and codimension, the set of functions satisfying (SAC1) is open in the C^1 topology, and those satisfying (USAC) are open in the C^2 topology (De Rosa, Tione, 2020).

Theorem: For arbitrary $n > d$, (AC1) is open in the C^1 topology on $C^1(\mathbb{G}(n,d))$.

Theorem: (SAC) and (AC) are neither open nor closed in the C^2-topology on $C^2(\mathbb{G}(n,n-1))$.

Future Work

- One line of investigation is to explore the relationships between these ellipcity conditions in general dimension; that is, for integrands defined on $\mathbb{G}(n,d)$.
- The ℓ^p norms have been used to construct a counterexample for topological properties of the ellipticity conditions. However, whether ℓ^p for $p \ne 2$ satisfies (AC) in higher dimension and codimension is still an open question.

References

[1] Antonio De Rosa and Riccardo Tione, *Regularity for graphs with bounded anisotropic mean curvature*, 2020.

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