Notions of Convexity and Ellipticity Conditions

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Abstract

In the study of regularity theory for anisotropic minimal surfaces, several ellipticity conditions have been posed, including the atomic condition (AC), the scalar atomic condition (SAC), and the uniform scalar atomic condition (USAC). Our work explores the relationships between these ellipticity conditions and searches for classes of integrands satisfying them. In particular, we construct a weaker condition (wAC) which is equivalent to convexity of C^1 integrands in codimension 1 and leads to a weakened rectifiability theorem. Analogously, we demonstrate equivalence between (USAC) and strong convexity, and consequently that (USAC) fails to hold for ℓ_p norms with $p > 2$.

Background

- Plateau's Problem
  Given a boundary γ, what is the surface with that boundary that minimizes the total area? What if we wish to minimize other quantities given the boundary γ?
- Varifold
  A d-dimensional Varifold V is a Radon measure on $\mathbb{R}^n \times \mathbb{G}(n,d)$.
  By disintegration of measure V can be decomposed as

$$V(dx, dT) = ||V||(dx) \otimes \mu_x(dT)$$

- Integrands and Energy
  Unless otherwise stated, assume all integrands are autonomous. Given a positive C^1 integrand $\Psi : \mathbb{G}(n,d) \to \mathbb{R}$, the anisotropic energy of a varifold V is defined as

$$\mathbf{\Psi}(V) := \int_{\mathbb{G}(n,d)} \Psi(T)d\mu(T)$$

  Given an integrand Ψ, its dual $\Psi^* : \mathbb{G}(n,d) \to \mathbb{R}$ is defined as

$$\Psi^*(P) := \Psi(\mathrm{id} - P) = \Psi(P^\perp) \quad \forall P \in \mathbb{G}(n,d)$$

  The *first variation* of the autonomous energy Ψ at the varifold V is given by

$$\delta_\Psi V(g) = \frac{d}{dt}\mathbf{\Psi}(\varphi_t^\# V, \Omega)\Big|_{t=0} = \int_{\mathbb{G}(n,d)} \langle B_\Psi(T), Dg(x)\rangle d\mu(T)$$

  where

$$\langle B_\Psi(T), L\rangle := \Psi(T)\langle T, L\rangle + \langle D\Psi(T), T^\perp LT + (T^\perp LT)^\top\rangle \quad \forall L \in \mathbb{R}^{n\times n}$$

  From this,

$$A(\mu) = \int B_\Psi(T)dV(T)$$

- Atomic Conditions
  Ψ satisfies the *atomic condition* (AC) if:
  - (AC1): $\dim\ker A(\mu) \le n-d$ for all $\mu \in \mathcal{P}(\mathbb{G}(n,d))$,
  - (AC2): if $\dim\ker A(\mu) = n-d$, then $\mu = \delta_{T_0}$ for some $T_0 \in \mathbb{G}(n,d)$.

  Ψ satisfies the *uniform scalar atomic condition* (USAC) if

$$\langle B_\Psi(T), B_{\Psi^*}(S)\rangle > C||T-S||^2 \quad \forall T \ne S \in \mathbb{G}(n,d)$$

- Convexity Notions
  Denote by G the 1-homogeneous extension of Ψ to $\mathbb{R}^n$ from the unit sphere, via the identification with $\mathbb{G}(n,d)$. We make use of the typical notion of convexity:

$$G(tv + (1-t)u) \le tG(v) + (1-t)G(u) \quad \forall u,v \in S^{n-1} \text{ and } t \in [0,1]$$

  G is *strongly convex* if

$$G(v) \ge G(u) + \nabla(G(u))^\top(v-u) + \frac{m}{2}||v-u||^2 \quad \forall u,v \in S^{n-1}$$

Characterization Theorem 1: (wAC) and Convexity

Theorem (De Philippis, De Rosa, Ghiraldin, 2016): In codimension 1, an integrand G satisfies the atomic condition if and only if the function G is strictly convex.

Definition: An integrand G satifies the *weak atomic condition* (wAC) if

- (AC1): $\dim\ker A(\mu) \le 1$ for all $\mu \in \mathcal{P}(\mathbb{G}(n,d))$,
- (wAC2): if $\dim\ker A(\mu) = 1$, then μ is supported on A_u for some $u \in S^{n-1}$ where

$$A_u = \{v \in S^{n-1} : \nabla G(v) = \pm\nabla G(u)\}$$

Theorem: In codimension 1, an integrand G satisfies (wAC) if and only if G is convex.

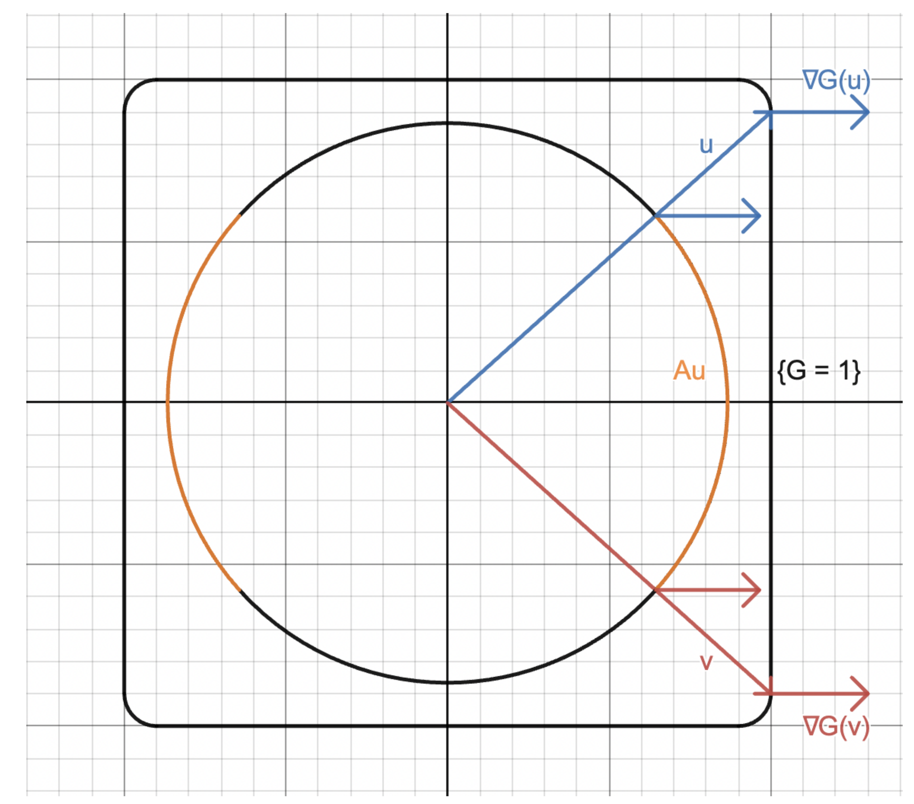
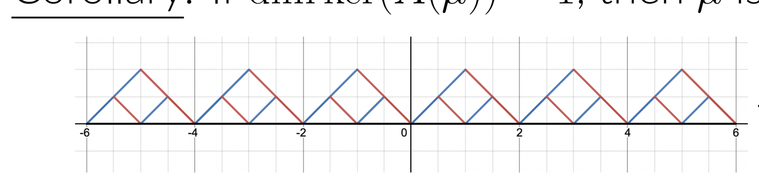


Figure 1. A level set for the mollified ℓ^∞ norm. A_u is concentrated on the right flat face, as the blue and red vectors have parallel gradients. In order for $\dim\ker A(\mu)$ to be one, the vectors must live in A_u.

Lemma: Let G be a C^1 convex function. Then,

$$A_u = \left\{\pm v \in S^{n-1} : G(u) = \langle\nabla G(v), u\rangle \text{ and } G(v) = \langle\nabla G(u), v\rangle\right\}.$$

Corollary: If $\dim\ker(A(\mu)) = 1$, then μ is supported on A_u for some $u \in \ker A(\mu)$.



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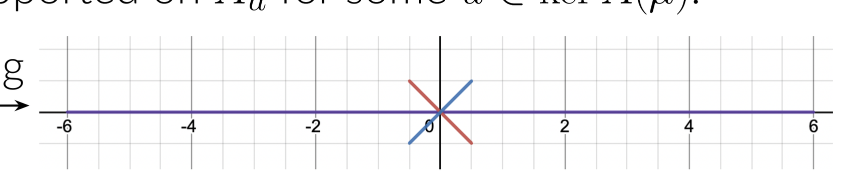

Figure 2. An example of a varifold with rectifiable support but not satisfying the atomic condition.

Figure 3. The tiling limit, a minimizer for Ψ (and also a stationary varifold with respect to Ψ), but $\mu \ne \delta_{T_0}$.

The (wAC) condition also leads to a weakened rectifiability theorem on varifolds:

Theorem: Let $G \in C^1(\Omega \times \mathbb{G}(n, n-1), \mathbb{R}_{>0})$ be a positive *non-autonomous* integrand. If G satisfies (wAC) at every $x \in \Omega$, then for every V with locally bounded first variation there exists a $n-1$-rectifiable set K such that

$$V_* = \theta\mathcal{H}^{n-1}\llcorner K \otimes \mu_x \quad \text{with } \mathrm{supp}(\mu_x) \subseteq A_{T_x K^\perp}$$

where

$$V_* := V\llcorner\{x \in \Omega : \theta^*(x,V) > 0\} \times \mathbb{G}(n, n-1).$$

Conversely, if every varifold V_* associated to V with locally bounded first variation has rectifiable support with $\mathrm{supp}(\mu_x) \in A_{T_x K^\perp}$, then Ψ satisfies (wAC).

Characterization Theorem 2: (USAC) and Strong Convexity

A 1-homogenous, even integrand $G : \mathbb{R}^n \to \mathbb{R}$ is strongly convex if there exists a constant $c > 0$ such that

$$G(u) - \langle dG(v), u\rangle \ge c|u-v|^2 \quad \forall u,v \in S^{n-1}$$

Theorem: In dimension or codimension 1, a C^2 integrand G satisfies (USAC) if and only if G is strongly convex.

Lemma: In codimension 1, G satisfying (USAC) is equivalent to there existing some $k > 0$ such that

$$G(u)G(v) - \langle dG(u), v\rangle\langle dG(v), u\rangle \ge k|u-v|^2|u+v|^2 \quad \forall u,v \in S^{n-1}$$

Corollary: In dimension and codimension 1, for any $p > 2$, the ℓ_p norm on $\mathbb{R}^n$ defined by

$$G(v) = \left(\sum_{i=1}^n |v_i|^p\right)^{1/p}$$

does *not* satisfy (USAC). We speak about the ℓ_p norm as a function on the Grassmannian manifold in dimension and codimension 1 via $\Psi(\mathrm{span}(v)) = G(v)$ and $\Phi(v^\perp) = \Psi^*(\mathrm{span}(v)) = G(v)$, respectively.

Dimension 1 Results

Theorem: Every positive integrand on $\mathbb{G}(n,1)$ satisfies (AC1). Hence, any varifold with locally bounded first variation has $\mathcal{H}^1$-rectifiable support.

This means any curve in $\mathbb{R}^n$ (for which $\delta_F(V)$ is a Radon measure) is $\mathcal{H}^1$-rectifiable (union of countably many Lipschitz graphs). The same is not true for codimension 1, and counterexamples exist in $G(3,2)$.

Theorem: Let G be a positive 1-homogeneous integrand on $\mathbb{R}^n$. Define $\Psi : \mathbb{G}(n, n-1) \to \mathbb{R}$ as $|v|\Psi(v^\perp) = G(v)$ (hence $|v|\Psi^*(v) = G(v)$). Then:

1. Ψ^* satisfying (AC) is equivalent to strict convexity of G.
2. Ψ^* satisfying (AC) is equivalent to Ψ satisfying (AC).
3. Ψ^* satisfying (wAC) is equivalent to convexity of G.

Future Work

- As it stands, (wAC) is only defined in dimension and codimension 1. We have a generalized notion, but it needs to be explored further.
- The relationship between ellipticity conditions and notions of convexity in higher dimension and codimension.

References

[1] Guido De Philippis, Antonio De Rosa, and Francesco Ghiraldin, *Rectifiability of varifolds with locally bounded first variation with respect to anisotropic surface energies*, 2016.

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