Homework 7. Due April 8

1. **(6 pts)** Consider a stochastic process $f(t,\omega)$ on $0 \le t \le T$ where ω indicates that f depends on a Brownian motion. Assume that:

   - $f(t,\omega)$ is independent of the increments of the Brownian motion $w(t,\omega)$ in the future, i.e., $f(t,\omega)$ is independent of $w(t+s,\omega) - w(t,\omega)$ for all $s > 0$.
   - $$\int_0^T E[f^2(s,\omega)]ds < \infty.$$

   Derive the properties below from the definition of the Ito stochastic integral (page 14 in `SDEs.pdf`).

   (a) If f is a deterministic function, i.e., $f(s,\omega) \equiv f(s)$, then
   $$\int_0^t f(s)dw(s,\omega) \sim N\left(0, \int_0^t f^2(s)ds\right).$$

   (b) For any $0 \le \tau \le t \le T$,
   $$E\left[\int_\tau^t f(s,\omega)dw(s,\omega)\right] = 0;$$

   (c) For any $0 \le \tau \le t \le T$,
   $$E\left[\int_\tau^t f(s,\omega)dw(s,\omega)\int_\tau^t g(s,\omega)dw(s,\omega)\right] = \int_\tau^t E[f(s,\omega)g(s,\omega)]ds.$$

2. **(4 pts)** Consider the SDE
   $$dX_t = b(X_t)dt + \sigma(X_t)dw, \quad X(0) = x \in \mathbb{R}^d, \quad t \in [0,T]. \tag{1}$$
   Show that
   $$\lim_{t\to s} E\left[\frac{X_t - X_s}{t-s} \mid X_s = x\right] = b(x,s) \tag{2}$$
   $$\lim_{t\to s} E\left[\frac{[X_t - X_s][X_t - X_s]^T}{t-s} \mid X_s = x\right] = \Sigma(x,s). \tag{3}$$
   The vector field $b(x,s)$ is called drift, and the matrix $\Sigma(x,s) = \sigma(x,s)\sigma(x,s)^T$ is called the diffusion matrix.

3. **(4 pts)** Find the analytical solution of the initial value problem
   $$dX_t = \left(\frac{b^2}{4} - X_t\right)dt + b\sqrt{X_t}dw, \quad X_0 = x > 0,$$
   where b is constant. Note that this process will stop as X_t reaches 0.
   Hint: make the variable change $Y = \sqrt{X}$ using the Ito formula.

4. **(6 pts)** The goal of this exercise is two-fold: (i) to see how one can approximate the Brownian motion with Fourier series with random coefficients, and (ii) to see that this approximation leads to Stratonovich's rather than Ito's SDE.

It was proven by Wiener that the Brownian motion on $[0,1]$ is the sum of a Fourier series

$$w(t) = \frac{a_0}{\sqrt{\pi}}t + \sqrt{\frac{2}{\pi}}\sum_{k=1}^{\infty}\frac{a_k}{k}\sin(kt), \tag{4}$$

where a_k, $k = 0, 1, 2, \ldots$, are independent Gaussian random variables with mean 0 and variance 1 (see this link).

Consider the geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dw_t. \tag{5}$$

The exact solution to (9) is given by

$$X_t = X_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma w_t}. \tag{6}$$

(a) Set $X_0 = 1$, $\mu = \sigma = 1$. Take the time interval $[0,1]$, split it to $N_t = 10^4$ subintervals, and generate $N_s = 10^3$ samples of the Brownian motion. Calculate the exact solutions for each of these samples using (6). Plot 10 solution samples. Calculate the mean and the variance of the N_s solution samples as functions of time and plot them. Superimpose these plots with the exact mean and variance given by

$$E[X_t] = X_0 e^{\mu t}, \quad \mathrm{Var}(X_t) = X_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right). \tag{7}$$

(b) Truncate the Fourier series in (4) at $k = 1000$. Use N_s sets of coefficients a_k and compute the mean and the variance of the truncated series at $t = 1$. You should get values close to 0 and 1 respectively.

(c) Approximate dw by the derivative of the truncated series multiplied by dt. Solve N_s ODEs obtained from (9) by plugging in this approximation to dw. Plot 10 solution samples. Then calculate the mean of the N_s solution samples as a function of t and plot it. Observe that this mean does not match the one in (7) but does match

$$X_0 e^{(\mu + \frac{1}{2}\sigma^2)t}. \tag{8}$$

The matter is that the ODE obtained by this smooth approximation to the Brownian motion approximates not Ito's by Stratonovich's SDE

$$dX_t = \mu X_t dt + \sigma X_t \circ dw_t. \tag{9}$$

In general, for a 1D Stratonovich's SDE

$$dX_t = b(X_t)dt + \sigma(X_t) \circ dw_t \tag{10}$$

the equivalent Ito's SDE is

$$dX_t = \left[b(X_t) + \frac{1}{2}\sigma'(X_t)\sigma(X_t)\right]dt + \sigma(X_t)dw_t. \tag{11}$$

A similar correction exists for $\mathbb{R}^d$ as well.