Turaev Torsion vs. Cohomology for 3-Manifolds with Boundary

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(Commutative) Reidemeister Torsion

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- $\tau^\varphi(X)$ is only defined up to multiplication by $\pm\varphi(H_1(X))$

Turaev's Refinements

In the 1980's, Turaev introduced refinements to torsions to remove the indeterminacies. For a 3-manifold M, these refinements are:

1. e a nonsingular vector field pointing outside M on ∂M, up to an equivalence relation (a *smooth Euler structure*).
2. ω a homology orientation of M.

The refined torsion is $\tau^{\varphi}(X, e, \omega)$

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- There is an (essentially unique) isomorphism $\Phi : Q(H_1(M)) \to \bigoplus_i \mathbb{F}_i$ where each $\mathbb{F}_i$ is a field
- The Turaev torsion is defined by:

$$\tau(M, e, \omega) = \Phi^{-1}\left(\bigoplus_i \tau^{\varphi_i}(M, e, \omega)\right) \in Q(H_1(M))$$

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- For links in S^3, the Turaev torsion is the Alexander polynomial

General Facts

If $\partial M \neq \varnothing$, $\chi(M) = 0$, and $b_1(M) \geq 2$, the Turaev torsion is in $\mathbb{Z}[H_1(M)] \subset Q(H_1(M))$, and in fact in the $b_1(M) - 2$ power of the augmentation ideal I of $\mathbb{Z}[H_1(M)]$. For a closed 3-manifold C with $b_1(C) \geq 3$, the torsion is in $I^{b_1(C)-3}$.

Cohomology Determinants

For a closed, oriented 3-manifold C, there is an alternate trilinear form on $H^1(C)$ given by cup products; $(x, y, z) \mapsto \langle x \cup y \cup z, [C] \rangle$. If $b_1(C) \geq 3$, such a form has an algebraic "determinant" which Turaev has related to torsion. If M is a 3-manifold with $\partial M \neq \varnothing$, however, one must be more careful.

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- The right way to do this is with the map $f_M : H^1(M, \partial M) \times H^1(M) \times H^1(M) \to \mathbb{Z}$ given by $(x, y, z) \mapsto \langle x \cup y \cup z, [M] \rangle$

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- If we also assume $n = b_1(M) \geq 2$, then this will have a "determinant"

The Construction

- Let $G = H_1(M)/\operatorname{Tors}(H_1(M))$; if $n = b_1(M)$, then G is a rank n free abelian group

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- Let $S(G) = \bigoplus_{r \geq 0} S^r$ denote the graded symmetric algebra on G
- For bases $\{a_i\}, \{b_j\}$ of $H^1(M), H^1(M, \partial M)$ respectively, let $\theta_{i,j} \in G$ be defined by:

$$\theta_{i,j}(z) = f_M(b_i, a_j, z) = \langle b_i \cup a_j \cup z, [M] \rangle$$

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For any bases a, b of $H^1(M), H^1(M, \partial M)$ respectively, if $n \geq 2$, there exists a $d(f_M, a, b) \in S^{n-2}$ such that

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and for any other bases a', b',

$$d(f_M, a', b') = [a'/a][b'/b] d(f_M, a, b)$$

- $[a'/a] \in \{\pm 1\}$ denotes the determinant of the change of basis matrix from a to a'

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- Given a homology orientation ω, one can choose a sign of d consistent with ω, denoted $\mathrm{Det}_\omega(f_M)$
- This is related to torsion via a graded map $q : S(G) \to \bigoplus_{\ell \geq 0} I^\ell / I^{\ell+1}$

The Main Theorem

Theorem

Let M be a compact, connected, oriented smooth 3-manifold with $\partial M \neq \varnothing$, $\chi(M) = 0$, and $n = b_1(M) \geq 2$. Then for any homology orientation ω and Euler structure e

$$\tau(M, e, \omega) \pmod{I^{n-1}} = q(\mathrm{Det}_\omega(f_M)) \in I^{n-2}/I^{n-1}$$

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- A similar determinant for the first nonvanishing Massey products is also related to torsion
- Gluing formulae for torsion and cohomology determinants can be used to relate all of these results to Turaev's results for closed 3-Manifolds.

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- Presumably, these results also say something about Seiberg-Witten; perhaps giving some sort of bounds.

References

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