MATH 436
EXAM 1
SEPTEMBER 20, 2007

SOLUTIONS

Instructions: Put one problem on each answer sheet. Only sign the honor pledge on the first sheet. Show all of your work.

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(1) For each item, say whether it is true or false. No justification is necessary (and no partial credit will be given). (4 points each)

(a) Every curve has a unit-speed reparametrization.
False. Every *regular* curve has a unit-speed reparametrization, but a singular curve does not.

(b) If $\boldsymbol{\gamma}\colon \mathbb{R} \to \mathbb{R}$ is a unit-speed curve, then $\dot{\boldsymbol{\gamma}}(s)$ is perpendicular to $\ddot{\boldsymbol{\gamma}}(s)$ for all $s \in \mathbb{R}$.
True. Proposition 1.2 in the textbook.

(c) The curvature of a space curve is only defined where the torsion is nonzero.
False. The torsion is defined where the curvature is nonzero.

(d) If $\boldsymbol{\gamma}\colon (0,1) \to \mathbb{R}^3$ is a regular curve with $\kappa(t) = 0$ for all $t \in (0,1)$, then the image of $\boldsymbol{\gamma}$ is part of a line.
True. If $\kappa \equiv 0$, then the tangent vector is constant, and the result follows from Proposition 1.1 in the textbook.

(e) The signed curvature of any simple closed curve is constant.
False. Only circles and lines have constant curvature.

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(2) (5 points each) State FOUR of the following.

(a) The definition of the arc-length of a curve $\boldsymbol{\delta}\colon (a,b) \to \mathbb{R}^n$.

(b) The definition of a singular point on a curve.

(c) The definition of the torsion of a unit-speed space curve with nonzero curvature.

(d) The Frenet-Serret equations

(e) The Isoperimetric Inequality

Solution: Find these in your textbook or notes.

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(3) (20 points) Suppose $\boldsymbol{\alpha}, \boldsymbol{\beta} \colon \mathbb{R} \to \mathbb{R}^2$ are two unit-speed curves parametrized by s. Let $\varphi(s)$ be the (counterclockwise) angle from the x–axis to $\dot{\boldsymbol{\alpha}}(s)$ for each $s \in \mathbb{R}$, and let $\vartheta(s)$ be the (counterclockwise) angle from the line $y = x$ to $\dot{\boldsymbol{\beta}}(s)$ for each $s \in \mathbb{R}$. Suppose that $\varphi(s) = \vartheta(s)$ for each $s \in \mathbb{R}$. Show that there is a rigid motion $M \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\boldsymbol{\alpha} = M \circ \boldsymbol{\beta}$. (You do not have to explicitly provide the rigid motion, just state why one exists.)

Proof. By Proposition 2.2 in the text, we know that the signed curvature of $\boldsymbol{\alpha}$, which we'll denote $\kappa_s^{\boldsymbol{\alpha}}$, satisfies

$$\kappa_s^{\boldsymbol{\alpha}} = \frac{d\varphi}{ds},$$

and the signed curvature of $\boldsymbol{\beta}$, which we'll denote $\kappa_s^{\boldsymbol{\beta}}$, satisfies

$$\kappa_s^{\boldsymbol{\beta}} = \frac{d\vartheta}{ds}.$$

But $\varphi(s) = \vartheta(s)$ for each $s \in \mathbb{R}$, so

$$\kappa_s^{\boldsymbol{\alpha}} = \kappa_s^{\boldsymbol{\beta}}.$$

But then the desired rigid motion exists by Theorem 2.1 in the text. □

(4) (20 points) Show that the ellipse

$$\boldsymbol{\varepsilon}(t) = (a\cos(t), b\sin(t)),$$

where a, b are positive real constants, is a simple closed curve (and give its period). Compute the area of the interior of the ellipse, and write an integral (but do not evaluate) which computes the length of the ellipse.

For Sale: The formula of the area of the interior of a simple closed curve, now at the low, low price of 3 points.

Proof. It is obvious that $\boldsymbol{\varepsilon}(t) = \boldsymbol{\varepsilon}(t + k2\pi)$, for any $k \in \mathbb{Z}$. Also, if $\boldsymbol{\varepsilon}(t) = \boldsymbol{\varepsilon}(t')$, then $\cos(t) = \cos(t')$ and $\sin(t) = \sin(t')$, which means that $t - t' = k2\pi$ for some $k \in \mathbb{Z}$. So $\boldsymbol{\varepsilon}$ is a simple closed curve with period 2π. Let $x(t) = a\cos(t)$ and $y(t) = b\sin(t)$, then

$$\begin{aligned}
\mathcal{A}\left(\operatorname{int}(\boldsymbol{\varepsilon})\right) &= \frac{1}{2}\int_0^{2\pi} (x\dot{y} - y\dot{x})\ dt \\
&= \frac{1}{2}\int_0^{2\pi} \left(ab\cos^2(t) + ab\sin^2(t)\right)\ dt \\
&= \pi ab
\end{aligned}$$

We can also easily compute

$$\dot{\boldsymbol{\varepsilon}} = (-a\sin(t), b\cos(t)),$$

so the length of $\boldsymbol{\varepsilon}$ is

$$\begin{aligned}
\ell\left(\boldsymbol{\varepsilon}\right) &= \int_0^{2\pi} \|\dot{\boldsymbol{\varepsilon}}\|\ dt \\
&= \int_0^{2\pi} \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}\ dt
\end{aligned}$$

□

(5) Let $\boldsymbol{\zeta}\colon \mathbb{R} \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\zeta}(t) = (t\cos(t), t\sin(t), 2t)\,.$$

(a) (10 points) Compute the curvature of $\boldsymbol{\zeta}$.
(b) (10 points) Compute the torsion of $\boldsymbol{\zeta}$.

You may find the following formulae useful:

$$\kappa = \frac{\left\| \ddot{\boldsymbol{\zeta}} \times \dot{\boldsymbol{\zeta}} \right\|}{\left\| \dot{\boldsymbol{\zeta}} \right\|^3}$$

$$\tau = \frac{\left(\dot{\boldsymbol{\zeta}} \times \ddot{\boldsymbol{\zeta}}\right) \cdot \dddot{\boldsymbol{\zeta}}}{\left\| \dot{\boldsymbol{\zeta}} \times \ddot{\boldsymbol{\zeta}} \right\|^2}$$

I'll save some time and just write the answers for all the parts:

$$\begin{aligned}
\boldsymbol{\zeta}(t) &= (t\cos(t), t\sin(t), 2t) \\
\dot{\boldsymbol{\zeta}}(t) &= (\cos(t) - t\sin(t), \sin(t) + t\cos(t), 2) \\
\ddot{\boldsymbol{\zeta}}(t) &= (-2\sin(t) - t\cos(t), 2\cos(t) - t\sin(t), 0) \\
\dddot{\boldsymbol{\zeta}}(t) &= (-3\cos(t) + t\sin(t), -3\sin(t) - t\cos(t), 0) \\
\left\| \dot{\boldsymbol{\zeta}} \right\| &= \sqrt{5 + t^2} \\
\left\| \ddot{\boldsymbol{\zeta}} \times \dot{\boldsymbol{\zeta}} \right\| &= \sqrt{t^4 + 20 + 8t^2} \\
\left(\dot{\boldsymbol{\zeta}} \times \ddot{\boldsymbol{\zeta}}\right) \cdot \dddot{\boldsymbol{\zeta}} &= 2t^2 + 12.
\end{aligned}$$

So now we can easily write

$$\kappa = \frac{\sqrt{t^4 + 20 + 8t^2}}{(\sqrt{5 + t^2})^3}$$

$$\tau = \frac{2t^2 + 12}{t^4 + 20 + 8t^2}$$