MATH 436
EXAM 1
SEPTEMBER 20, 2007

Instructions: Put one problem on each answer sheet. Only sign the honor pledge on the first sheet. Show all of your work.

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(1) For each item, say whether it is true or false. No justification is necessary (and no partial credit will be given). (4 points each)
  (a) Every curve has a unit-speed reparametrization.
  (b) If $\boldsymbol{\gamma}\colon \mathbb{R} \to \mathbb{R}$ is a unit-speed curve, then $\dot{\boldsymbol{\gamma}}(s)$ is perpendicular to $\ddot{\boldsymbol{\gamma}}(s)$ for all $s \in \mathbb{R}$.
  (c) The curvature of a space curve is only defined where the torsion is nonzero.
  (d) If $\boldsymbol{\gamma}\colon (0,1) \to \mathbb{R}^3$ is a regular curve with $\kappa(t) = 0$ for all $t \in (0,1)$, then the image of $\boldsymbol{\gamma}$ is part of a line.
  (e) The signed curvature of any simple closed curve is constant.

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(2) (5 points each) State FOUR of the following.
  (a) The definition of the arc-length of a curve $\boldsymbol{\delta}\colon (a,b) \to \mathbb{R}^n$.
  (b) The definition of a singular point on a curve.
  (c) The definition of the torsion of a unit-speed space curve with nonzero curvature.
  (d) The Frenet-Serret equations
  (e) The Isoperimetric Inequality

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(3) (20 points) Suppose $\boldsymbol{\alpha}, \boldsymbol{\beta}\colon \mathbb{R} \to \mathbb{R}^2$ are two unit-speed curves parametrized by s. Let $\varphi(s)$ be the (counterclockwise) angle from the x–axis to $\dot{\boldsymbol{\alpha}}(s)$ for each $s \in \mathbb{R}$, and let $\vartheta(s)$ be the (counterclockwise) angle from the line $y = x$ to $\dot{\boldsymbol{\beta}}(s)$ for each $s \in \mathbb{R}$. Suppose that $\varphi(s) = \vartheta(s)$ for each $s \in \mathbb{R}$. Show that there is a rigid motion $M\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\boldsymbol{\alpha} = M \circ \boldsymbol{\beta}$. (You do not have to explicitly provide the rigid motion, just state why one exists.)

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(4) (20 points) Show that the ellipse

$$\boldsymbol{\varepsilon}(t) = (a\cos(t), b\sin(t)),$$

where a, b are positive real constants, is a simple closed curve (and give its period). Compute the area of the interior of the ellipse, and write an integral (but do not evaluate) which computes the length of the ellipse.
For Sale: The formula of the area of the interior of a simple closed curve, now at the low, low price of 3 points.

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(5) Let $\boldsymbol{\zeta}\colon \mathbb{R} \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\zeta}(t) = (t\cos(t), t\sin(t), 2t).$$

  (a) (10 points) Compute the curvature of $\boldsymbol{\zeta}$.
  (b) (10 points) Compute the torsion of $\boldsymbol{\zeta}$.

You may find the following formulae useful:

$$\kappa = \frac{\left\| \ddot{\boldsymbol{\zeta}} \times \dot{\boldsymbol{\zeta}} \right\|}{\left\| \dot{\boldsymbol{\zeta}} \right\|^3}$$

$$\tau = \frac{\left(\dot{\boldsymbol{\zeta}} \times \ddot{\boldsymbol{\zeta}}\right) \cdot \dddot{\boldsymbol{\zeta}}}{\left\| \dot{\boldsymbol{\zeta}} \times \ddot{\boldsymbol{\zeta}} \right\|^2}$$
