MATH 436
EXAM 2
OCTOBER 16, 2007

SOLUTIONS

Instructions: Put one problem on each answer sheet. Only sign the honor pledge on the first sheet. Show all of your work.

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(1) For each item, say whether it is true or false. No justification is necessary (and no partial credit will be given). (4 points each)

(a) Every subset of $\mathbb{R}^2$ is a smooth surface.
False. Counterexamples abound, like say a single point.

(b) The transition maps of a smooth surface are diffeomorphisms.
True. This follows from Propostion 4.1 in Pressley.

(c) The first fundamental form of a surface patch is invariant under reparametrization.
False. I showed how first fundamental form changes under reparametrization in class; it is also in Exercise 5.4 in Pressley.

(d) Every conformal map is also an isometry.
False. The converse, however, is true. Conformal maps give proportional first fundamental forms, isometries require equal first fundamental forms.

(e) The sum of the angles of a triangle on the unit sphere S^2 whose sides are great circles is greater than π.
True. This is required by the area formula of such a triangle.

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(2) (5 points each) State FOUR of the following.

(a) The definition of an open subset of $\mathbb{R}^n$.

(b) The definition of a regular surface patch.

(c) The definition of the normal vector to a regular surface patch.

(d) The definition of the first fundamental form of a regular surface patch (you may simply define the three functions).

(e) If $f\colon S_1 \to S_2$ is a diffeomorphism, state the relationship that must hold between the first fundamental forms of a regular patch $\boldsymbol{\sigma}_1$ of S_1 to the regular patch $\boldsymbol{\sigma}_2 = f \circ \boldsymbol{\sigma}_1$ of S_2 (for each regular patch $\boldsymbol{\sigma}_1$ of S_1) in order to know that f is equiareal.

Solution: See your text or notes for these. □

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(3) (20 points) **Compute** (don't just state the answer) the first fundamental forms for each surface patch of a smooth atlas for the unit sphere S^2 (that is, an atlas consisting of all regular surface patches). Carefully define the atlas, including the domain of each patch. If you want to use a smooth atlas that we've never discussed, then you should show that the patches are regular. On the other hand, if you are using a smooth atlas that we've used in class (I recall three), then you don't need to show regularity.

Solution: I'll do the three atlases we've covered. The first I'll do is the 6 hemispheres,

$$\begin{aligned}\boldsymbol{\sigma}^{z,\pm}(u,v) &= (u, v, \pm\sqrt{1-u^2-v^2})\\ \boldsymbol{\sigma}^{x,\pm}(u,v) &= (\pm\sqrt{1-u^2-v^2}, u, v)\\ \boldsymbol{\sigma}^{y,\pm}(u,v) &= (v, \pm\sqrt{1-u^2-v^2}, u).\end{aligned}$$

The only one that might be somewhat surprising is $\boldsymbol{\sigma}^{y,\pm}$; the reason for the ordering of the u and the v is for this to be an oriented atlas. You can certainly use any order for u and v; I just wanted my atlas to be oriented. Now some derivatives:

$$\begin{aligned}\boldsymbol{\sigma}_u^{z,\pm}(u,v) &= \left(1, 0, \mp\frac{u}{\sqrt{1-u^2-v^2}}\right)\\ \boldsymbol{\sigma}_v^{z,\pm}(u,v) &= \left(0, 1, \mp\frac{v}{\sqrt{1-u^2-v^2}}\right).\end{aligned}$$

We can easily compute the derivatives for the other 4 charts similarly. So

$$\begin{aligned}E^{z,\pm} &= 1 + \frac{u^2}{1-u^2-v^2}\\ &= \frac{1-v^2}{1-u^2-v^2}\\ F^{z,\pm} &= \frac{uv}{1-u^2-v^2}\\ G^{z,\pm} &= \frac{1-u^2}{1-u^2-v^2}.\end{aligned}$$

So the first fundamental forms for these patches are

$$ds^2_{z,\pm} = \left(\frac{1-v^2}{1-u^2-v^2}\right)du^2 + \left(\frac{2uv}{1-u^2-v^2}\right)du dv + \left(\frac{1-u^2}{1-u^2-v^2}\right)dv^2.$$

Now it is not difficult to see that we'll get the exact same first fundamental form for all of the other charts too, since each other chart is simply a cyclic permutation of the coordinates of these two charts, which won't change the dot products of the derivatives. So for each patch $\boldsymbol{\sigma}$ in this atlas we have

$$ds^2 = \left(\frac{1-v^2}{1-u^2-v^2}\right)du^2 + \left(\frac{2uv}{1-u^2-v^2}\right)du dv + \left(\frac{1-u^2}{1-u^2-v^2}\right)dv^2.$$

Now we can do the one that relies on spherical coordinates. This will require two charts:

$$\begin{aligned}\boldsymbol{\sigma}^1(\theta,\varphi) &= (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))\\ \boldsymbol{\sigma}^2(\theta,\varphi) &= (\sin(\theta)\sin(\varphi), \cos(\theta), \sin(\theta)\cos(\varphi)).\end{aligned}$$

The domain of $\boldsymbol{\sigma}^1$ is $0 < \theta < \pi, 0 < \varphi < 2\pi$, which means that it excludes the north and south pole, and the half of the great circle connecting the north and south pole and intersecting the positive x–axis. The domain of $\boldsymbol{\sigma}^2$ is $0 < \theta < \pi, -\frac{\pi}{2} < \varphi < \frac{3\pi}{2}$, which means that it excludes the poles on the y–axis, and the half of the great circle connecting those poles and intersecting the negative x–axis. However, as in the last problem, it is clear that the first fundamental forms for the two patches are the same, since they are just a coordinate-permutation away from being the same, which won't change the dot products of derivatives. So for the first patch:

$$\boldsymbol{\sigma}^1_\theta = (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), -\sin(\theta))$$
$$\boldsymbol{\sigma}^1_\varphi = (-\sin(\theta)\sin(\varphi), \sin(\theta)\cos(\varphi), 0).$$

So now the components of the first fundamental form for this patch are

$$E^1 = 1$$
$$F^1 = 0$$
$$G^1 = \sin^2(\theta).$$

So the first fundamental forms for both of these patches both are given by

$$ds^2 = d\theta^2 + \sin^2(\theta)d\varphi^2.$$

Now for stereographic projection, which is probably the most difficult. The two patches are

$$\boldsymbol{\sigma}_{NP}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right),$$
$$\boldsymbol{\sigma}_{SP}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}\right).$$

Rather than boring you with the details,

$$E_{NP} = \frac{4}{(u^2+v^2+1)^2}$$
$$F_{NP} = 0$$
$$G_{NP} = \frac{4}{(u^2+v^2+1)^2},$$

so the first fundamental form for this patch is

$$ds^2_{NP} = \left(\frac{4}{(u^2+v^2+1)^2}\right)du^2 + \left(\frac{4}{(u^2+v^2+1)^2}\right)dv^2.$$

Similarly,

$$ds^2_{SP} = \left(\frac{4}{(u^2+v^2+1)^2}\right)du^2 + \left(\frac{4}{(u^2+v^2+1)^2}\right)dv^2.$$

$\square$

(4) (a) (10 points) Prove that any smooth surface consisting of a single (regular) surface patch is orientable.

Proof. The definition of orientable is possessing a smooth atlas where all transition functions have positive Jacobian determinants. If there is only one regular surface patch, the only transition function is the identity map, which has Jacobian determinant 1, which is positive. (Recall transition maps are things like $\boldsymbol{\sigma}_2^{-1} \circ \boldsymbol{\sigma}_1$, where $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ are patches.) □

(b) (5 points) Show that any open subset of a plane is an orientable surface.

Proof. Recall that a plane has a single surface patch, and any open subset of $\mathbb{R}^2$ will have a single patch, so any open subset of a plane can be given an atlas consisting of a single patch, hence is orientable by part (a). □

(c) (5 points) Show that the Möbius band cannot be covered with a single regular surface patch.

Proof. The Möbius band is not orientable, so it can't be covered by a single patch by part (a). □

(5) (a) (10 points) Derive the formula for the (cosine of) the angle between two curves on a regular surface patch $\boldsymbol{\sigma}\colon U \to \mathbb{R}^3$ given by $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ and $\widetilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\sigma}(\widetilde{u}(t), \widetilde{v}(t))$ in terms of the components E, F, and G of the first fundamental form of a surface patch $\boldsymbol{\sigma}$. You will need to know the definition of the angle between two curves, and how to obtain that angle using dot products.

Solution: See the text or your notes. □

(b) (10 points) Let $U = (-\pi, \pi) \times \mathbb{R} \subset \mathbb{R}^2$, and $\boldsymbol{\sigma}\colon U \to \mathbb{R}^3$ be the map

$$\boldsymbol{\sigma}(u, v) = (\cos(u), \sin(u), v) .$$

Compute the angle between the curves $\boldsymbol{\gamma}, \widetilde{\boldsymbol{\gamma}}$ at the point $(1, 0, 0)$ on $\boldsymbol{\sigma}(U)$, where

$$\boldsymbol{\gamma}(t) = (\cos(t), \sin(t), t)$$
$$\widetilde{\boldsymbol{\gamma}}(t) = (1, 0, t) .$$

Hint: the formula you derived in part (a) might not be the easiest way to compute this.

Solution: It is easiest to completely ignore the surface patch, and just compute the angle between the tangent vectors at $t = 0$, where the curves intersect.

$$\dot{\boldsymbol{\gamma}}(t) = (-\sin(t), \cos(t), 1)$$
$$\dot{\widetilde{\boldsymbol{\gamma}}}(t) = (0, 0, 1) .$$

So at $t = 0$, we are looking at the angle between $(0, 1, 1)$ and $(0, 0, 1)$. This angle is clearly $\dfrac{\pi}{4}$. □