MATH 436
EXAM 2
OCTOBER 16, 2007

Instructions: Put one problem on each answer sheet. Only sign the honor pledge on the first sheet. Show all of your work.

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(1) For each item, say whether it is true or false. No justification is necessary (and no partial credit will be given). (4 points each)
  (a) Every subset of $\mathbb{R}^2$ is a smooth surface.
  (b) The transition maps of a smooth surface are diffeomorphisms.
  (c) The first fundamental form of a surface patch is invariant under reparametrization.
  (d) Every conformal map is also an isometry.
  (e) The sum of the angles of a triangle on the unit sphere S^2 whose sides are great circles is greater than π.

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(2) (5 points each) State FOUR of the following.
  (a) The definition of an open subset of $\mathbb{R}^n$.
  (b) The definition of a regular surface patch.
  (c) The definition of the normal vector to a regular surface patch.
  (d) The definition of the first fundamental form of a regular surface patch (you may simply define the three functions).
  (e) If $f\colon S_1 \to S_2$ is a diffeomorphism, state the relationship that must hold between the first fundamental forms of a regular patch $\boldsymbol{\sigma}_1$ of S_1 to the regular patch $\boldsymbol{\sigma}_2 = f \circ \boldsymbol{\sigma}_1$ of S_2 (for each regular patch $\boldsymbol{\sigma}_1$ of S_1) in order to know that f is equiareal.

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(3) (20 points) **Compute** (don't just state the answer) the first fundamental forms for each surface patch of a smooth atlas for the unit sphere S^2 (that is, an atlas consisting of all regular surface patches). Carefully define the atlas, including the domain of each patch. If you want to use a smooth atlas that we've never discussed, then you should show that the patches are regular. On the other hand, if you are using a smooth atlas that we've used in class (I recall three), then you don't need to show regularity.

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(4) (a) (10 points) Prove that any smooth surface consisting of a single (regular) surface patch is orientable.
  (b) (5 points) Show that any open subset of a plane is an orientable surface.
  (c) (5 points) Show that the Möbius band cannot be covered with a single regular surface patch.

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(5) (a) (10 points) Derive the formula for the (cosine of) the angle between two curves on a regular surface patch $\boldsymbol{\sigma}\colon U \to \mathbb{R}^3$ given by $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ and $\widetilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\sigma}(\widetilde{u}(t), \widetilde{v}(t))$ in terms of the components E, F, and G of the first fundamental form of a surface patch $\boldsymbol{\sigma}$. You will need to know the definition of the angle between two curves, and how to obtain that angle using dot products.
  (b) (10 points) Let $U = (-\pi, \pi) \times \mathbb{R} \subset \mathbb{R}^2$, and $\boldsymbol{\sigma}\colon U \to \mathbb{R}^3$ be the map

$$\boldsymbol{\sigma}(u, v) = (\cos(u), \sin(u), v)\,.$$

  Compute the angle between the curves $\boldsymbol{\gamma}, \widetilde{\boldsymbol{\gamma}}$ at the point $(1, 0, 0)$ on $\boldsymbol{\sigma}(U)$, where

$$\boldsymbol{\gamma}(t) = (\cos(t), \sin(t), t)$$
$$\widetilde{\boldsymbol{\gamma}}(t) = (1, 0, t)$$

  Hint: the formula you derived in part (a) might not be the easiest way to compute this.
