MATH 436
EXAM 3
NOVEMBER 13, 2007

Instructions: Put one problem on each answer sheet. Only sign the honor pledge on the first sheet. Show all of your work.

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(1) For each item, say whether it is true or false. No justification is necessary (and no partial credit will be given). (4 points each)

(a) The matrix $\mathcal{F}_{II}$ is always nonsingular.
False. A plane is a counterexample. However, $\mathcal{F}_I$ is always nonsingular.

(b) The principal vectors corresponding to distinct principal curvatures at a non-umbilic point are orthogonal.
True. This was part of an important theorem.

(c) The Gauss map on a flat surface is conformal.
False. A plane is a counterexample. This is true if you replace "flat" with "minimal."

(d) If $\boldsymbol{\gamma}$ is a simple closed curve on a minimal surface patch $\boldsymbol{\sigma}$, then the area $\mathcal{A}_{\boldsymbol{\sigma}}(\text{int}(\gamma))$ is minimal among all areas of surfaces bounded by $\boldsymbol{\gamma}$.
False. This is the converse of basically the only theorem about minimal surfaces that we know.

(e) If the Gaussian curvature K is zero at a point x in a surface S, then x is a planar point.
False. It could also be parabolic.

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(2) (5 points each) State FOUR of the following.

(a) The definition of the normal and geodesic curvatures of a curve $\boldsymbol{\gamma}$ on a patch $\boldsymbol{\sigma}$.

(b) The definition of the second fundamental form of a surface patch $\boldsymbol{\sigma}$ (all three functions).

(c) The definition of the principal curvatures of a surface patch $\boldsymbol{\sigma}$.

(d) The relationship between the normal curvature of a curve on a surface patch to the principal curvatures of the patch (this should also involve an angle, and you should explain how that angle is obtained).

(e) The definitions of the matrices $\mathcal{F}_I, \mathcal{F}_{II}$, and $\mathcal{W}$.

See your text or notes.

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(3) The parts in this problem are completely independent.

(a) (10 points) Suppose $\boldsymbol{\sigma}$ is a (regular) surface patch, and $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ is a unit-speed curve on $\boldsymbol{\sigma}$. Derive the formula for the normal curvature of $\boldsymbol{\gamma}$ in terms of the second fundamental form of $\boldsymbol{\sigma}$.

Solution:

$$\begin{aligned}
\kappa_n &= \ddot{\boldsymbol{\gamma}} \cdot \boldsymbol{N} \\
&= \left(\frac{d}{dt}\dot{\boldsymbol{\gamma}}\right) \cdot \boldsymbol{N} \\
&= \left(\frac{d}{dt}\left(\boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}\right)\right) \cdot \boldsymbol{N} \\
&= \left[\left(\boldsymbol{\sigma}_{uu}\dot{u} + \boldsymbol{\sigma}_{uv}\dot{v}\right)\dot{u} + \boldsymbol{\sigma}_u \ddot{u} + \left(\boldsymbol{\sigma}_{vu}\dot{u} + \boldsymbol{\sigma}_{vv}\dot{v}\right)\dot{v} + \boldsymbol{\sigma}_v \ddot{v}\right] \cdot \boldsymbol{N} \\
&= L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2.
\end{aligned}$$

The last equality holds since $\boldsymbol{N}$ is perpendicular to both $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ and by the definitions of L, M, and N. □

(b) (10 points) Recall that a curve $\mathcal{C}$ on a surface S is a *line of curvature* of S if the tangent vector to $\mathcal{C}$ is a principal vector at each point of $\mathcal{C}$. Suppose that S is a surface such that *every* curve on S is a line of curvature, and suppose that $\boldsymbol{\gamma}$ is a curve on S whose normal curvature vanishes, i.e. $\kappa_n = 0$, at some point on $\boldsymbol{\gamma}$. Show that S is part of a plane.

Proof. Since every curve is a line of curvature, every tangent vector is a principal vector. This means that every point on S is an umbilic, so S is either part of a plane or part of a sphere. But if there is a curve with $\kappa_n = 0$ at any point, then at that point we have $\kappa_n = \kappa_1 = 0$, and the point is umbilic so $\kappa_2 = 0$ also, hence S isn't part of a sphere. So S is part of a plane. □

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(4) The parts in this problem are completely independent.

(a) (10 points) Suppose that $\boldsymbol{\sigma}(u, v)$ is a flat surface patch, and let $\boldsymbol{N}$, as usual, denote its unit normal vector. Show that $\boldsymbol{N}_u$ and $\boldsymbol{N}_v$ are linearly *dependent* at each point on $\boldsymbol{\sigma}$.

Proof. Since $\boldsymbol{N}_u = a\boldsymbol{\sigma}_u + b\boldsymbol{\sigma}_v$ and $\boldsymbol{N}_v = c\boldsymbol{\sigma}_u + d\boldsymbol{\sigma}_v$, where $\mathcal{W} = -\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is the Weingarten matrix, it is a simple computation that

$$\boldsymbol{N}_u \times \boldsymbol{N}_v = \det(\mathcal{W})\, \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v.$$

But the principal curvatures are the eigenvalues of $\mathcal{W}$, so $\det(\mathcal{W}) = K$, the Gaussian curvature. This means that if $K = 0$ at each point on the surface, then

$$\boldsymbol{N}_u \times \boldsymbol{N}_v = 0$$

at each point, hence $\boldsymbol{N}_u$ and $\boldsymbol{N}_v$ are linearly dependent. □

(b) (10 points) Give an example of two surface patches that have the same first fundamental form and different second fundamental forms. Yes, you do have to compute the forms to get credit.

Solution: I don't need credit, so I'll just state that the cylinder

$$\boldsymbol{\sigma}_1(u, v) = (\cos(u), \sin(u), v),$$

and the plane

$$\boldsymbol{\sigma}_2(u, v) = (u, v, 0)$$

have the same first fundamental form $(du^2 + dv^2)$, and different second fundamental forms (the first is $-du^2$ and the second is 0). □

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(5) (a) (5 points) Show that the Gaussian curvature K of a minimal surface satisfies $K \leq 0$ at each point of the surface.

Proof. Since $H = 0$, if κ_1 and κ_2 are the principal curvatures, then $\kappa_1 + \kappa_2 = 0$. Then

$$\begin{aligned} \kappa_1 &= -\kappa_2 \\ \kappa_1 \kappa_2 &= -\kappa_2^2 \\ &\leq 0. \end{aligned}$$

□

(b) (5 points) Use part (a) to show that there are no compact minimal surfaces.

Proof. We know that each compact surface has (at least) on point with positive Gaussian curvature. But every point on a minimal surface has nonpositive Gaussian curvature. □

(c) (10 points) Let $U = \left\{(u, v) \in \mathbb{R}^2 \;\middle|\; -\frac{\pi}{2} < u < \frac{\pi}{2}, -\frac{\pi}{2} < v < \frac{\pi}{2}\right\}$, and let $\boldsymbol{\sigma}\colon U \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\sigma}(u, v) = \left(u, v, \ln\left(\frac{\cos(v)}{\cos(u)}\right)\right).$$

Show that $\boldsymbol{\sigma}$ is a minimal surface (this is independent of parts (a) and (b)). You may find one of the following formulae useful:

$$H = \frac{LG - 2MF + NE}{2\left(EG - F^2\right)}$$

$$K = \frac{LN - M^2}{EG - F^2}.$$

Proof. We just want to compute H. These are fairly straightforward.

$$\begin{aligned} E &= \sec^2(u) \\ F &= -\tan(u)\tan(v) \\ G &= \sec^2(v) \\ L &= \frac{\sec^2(u)}{\sqrt{1 + \tan^2(u) + \tan^2(v)}} \\ M &= 0 \\ N &= -\frac{\sec^2(v)}{\sqrt{1 + \tan^2(u) + \tan^2(v)}}. \end{aligned}$$

Plugging these in to the equation for H, we can easily see that $H = 0$. □