

MATH 436
EXAM 3
NOVEMBER 13, 2007

Instructions: Put one problem on each answer sheet. Only sign the honor pledge on the first sheet. Show all of your work.

- (1) For each item, say whether it is true or false. No justification is necessary (and no partial credit will be given). (4 points each)
- (a) The matrix \mathcal{F}_{II} is always nonsingular.
 - (b) The principal vectors corresponding to distinct principal curvatures at a non-umbilic point are orthogonal.
 - (c) The Gauss map on a flat surface is conformal.
 - (d) If γ is a simple closed curve on a minimal surface patch σ , then the area $\mathcal{A}_\sigma(\text{int}(\gamma))$ is minimal among all areas of surfaces bounded by γ .
 - (e) If the Gaussian curvature K is zero at a point x in a surface S , then x is a planar point.
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- (2) (5 points each) State FOUR of the following.
- (a) The definition of the normal and geodesic curvatures of a curve γ on a patch σ .
 - (b) The definition of the second fundamental form of a surface patch σ (all three functions).
 - (c) The definition of the principal curvatures of a surface patch σ .
 - (d) The relationship between the normal curvature of a curve on a surface patch to the principal curvatures of the patch (this should also involve an angle, and you should explain how that angle is obtained).
 - (e) The definitions of the matrices \mathcal{F}_I , \mathcal{F}_{II} , and \mathcal{W} .
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- (3) The parts in this problem are completely independent.
- (a) (10 points) Suppose σ is a (regular) surface patch, and $\gamma(t) = \sigma(u(t), v(t))$ is a unit-speed curve on σ . Derive the formula for the normal curvature of γ in terms of the second fundamental form of σ .
 - (b) (10 points) Recall that a curve \mathcal{C} on a surface S is a *line of curvature* of S if the tangent vector to \mathcal{C} is a principal vector at each point of \mathcal{C} . Suppose that S is a surface such that *every* curve on S is a line of curvature, and suppose that γ is a curve on S whose normal curvature vanishes, i.e. $\kappa_n = 0$, at some point on γ . Show that S is part of a plane.
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- (4) The parts in this problem are completely independent.
- (a) (10 points) Suppose that $\sigma(u, v)$ is a flat surface patch, and let \mathbf{N} , as usual, denote its unit normal vector. Show that \mathbf{N}_u and \mathbf{N}_v are linearly *dependent* at each point on σ .
 - (b) (10 points) Give an example of two surface patches that have the same first fundamental form and different second fundamental forms. Yes, you do have to compute the forms to get credit.
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- (5) (a) (5 points) Show that the Gaussian curvature K of a minimal surface satisfies $K \leq 0$ at each point of the surface.
- (b) (5 points) Use part (a) to show that there are no compact minimal surfaces.
- (c) (10 points) Let $U = \{(u, v) \in \mathbb{R}^2 \mid -\frac{\pi}{2} < u < \frac{\pi}{2}, -\frac{\pi}{2} < v < \frac{\pi}{2}\}$, and let $\sigma: U \rightarrow \mathbb{R}^3$ be defined by

$$\sigma(u, v) = \left(u, v, \ln \left(\frac{\cos(v)}{\cos(u)} \right) \right).$$

Show that σ is a minimal surface (this is independent of parts (a) and (b)). You may find one of the following formulae useful:

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}$$

$$K = \frac{LN - M^2}{EG - F^2}.$$
