MATH 436
HOMEWORK 1
DUE SEPTEMBER 11, 2007

SOLUTIONS

(1) Find a parametrized curve $\alpha\colon \mathbb{R} \to \mathbb{R}^2$ whose image is the circle

$$\left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1\right\},$$

with $\alpha(t)$ running clockwise around the circle, and $\alpha(0) = (0, 1)$.

Solution: There are many possible solutions. The first that occurs to me is

$$\alpha(t) = (\sin(t), \cos(t)) .$$

□

(2) Let $\beta\colon \mathbb{R} \to \mathbb{R}^3$ be defined by

$$\beta(t) = (3\cos(t), 3\sin(t), 4t) .$$

Compute the arc-length of β between 0 and t for any t.

Solution: We can compute

$$\begin{aligned} \dot\beta(t) &= (-3\sin(t), 3\cos(t), 4) \\ \|\dot\beta(t)\|^2 &= [-3\sin(t)]^2 + [3\cos(t)]^2 + 4^2 \\ &= 25 \end{aligned}$$

So $\|\dot\beta(t)\| = 5$ for all t, hence

$$\begin{aligned} s(t) &= \int_0^t \|\dot\beta(u)\|\, du \\ &= 5t \end{aligned}$$

□

Date: September 4, 2007.

(3) Let $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ be defined by

$$\gamma(t) = (\cosh(t), \sin(t), \cos(t)).$$

Find a unit-speed reparametrization of γ.

Solution: First we need to compute the arc-length function.

$$\begin{aligned}
\dot\gamma(t) &= (\sinh(t), \cos(t), -\sin(t)) \\
\|\dot\gamma(t)\|^2 &= \sinh^2(t) + \cos^2(t) + \sin^2(t) \\
&= \cosh^2(t) \\
s(t) &= \int_0^t \|\dot\gamma(u)\|\ du \\
&= \int_0^t \cosh(u)\ du \\
&= \sinh(t).
\end{aligned}$$

So $s^{-1}(t)$ is $\operatorname{arsinh}(t)$, and a unit-speed reparametrization of γ is

$$\widetilde{\gamma}(t) = (\cosh(\operatorname{arsinh}(t)), \sin(\operatorname{arsinh}(t)), \cos(\operatorname{arsinh}(t))).$$

Let's check that this actually is a unit speed curve. To do this, we'll need to recall

$$\frac{d}{dt}\operatorname{arsinh}(t) = \frac{1}{\sqrt{1+t^2}}.$$

So let's compute $\|\dot{\widetilde{\gamma}}(t)\|^2$:

$$\begin{aligned}
\dot{\widetilde{\gamma}}(t) &= \left(\sinh(\operatorname{arsinh}(t))\frac{1}{\sqrt{1+t^2}}, \cos(\operatorname{arsinh}(t))\frac{1}{\sqrt{1+t^2}}, -\sin(\operatorname{arsinh}(t))\frac{1}{\sqrt{1+t^2}}\right) \\
&= \frac{1}{\sqrt{1+t^2}}\left(t, \cos(\operatorname{arsinh}(t)), -\sin(\operatorname{arsinh}(t))\right)
\end{aligned}$$

$$\begin{aligned}
\|\dot{\widetilde{\gamma}}(t)\|^2 &= \frac{1}{1+t^2}\left[t^2 + \cos^2(\operatorname{arsinh}(t)) + \sin^2(\operatorname{arsinh}(t))\right] \\
&= \frac{1}{1+t^2}\left[t^2+1\right] \\
&= 1
\end{aligned}$$

□

(4) Let $\delta\colon (a,b) \to \mathbb{R}^3$ be a curve, and $v \in \mathbb{R}^3$ a fixed vector. Suppose that $\delta(t_0)$ is perpendicular to v for some $t_0 \in (a,b)$, and that for all $t \in (a,b)$, $\dot{\delta}(t)$ is perpendicular to v. Show that $\delta(t)$ is perpendicular to v for all $t \in (a,b)$.

Proof. Recall
$$\frac{d}{dt}(a \cdot b) = \dot{a} \cdot b + a \cdot \dot{b},$$
so using this and the fact that v is fixed hence $\dot{v} = 0$,
$$\begin{aligned} \frac{d}{dt}(\delta \cdot v) &= \dot{\delta} \cdot v + \delta \cdot \dot{v} \\ &= \dot{\delta} \cdot v \\ &= 0. \end{aligned}$$
So $\delta \cdot v$ is constant as a function of t. But $\delta(t_0) \cdot v = 0$, so $\delta(t) \cdot v = 0$ for all $t \in (a,b)$. □