MATH 436
HOMEWORK 10
DUE DECEMBER 11, 2007

SOLUTIONS

(1) Let $\boldsymbol{\sigma}\colon (0,2\pi)\times(0,2\pi)\to\mathbb{R}^3$ be the surface patch defined by

$$\boldsymbol{\sigma}(u,v) = \left((\cos(u)+3)\cos(v), (\cos(u)+3)\sin(v), \sin(u)\right).$$

Compute the Christoffel symbols Γ^k_{ij} for this patch.

Solution: Recall from Homework 5 that this is a patch for a torus, and it has first fundamental form

$$E = 1$$
$$F = 0$$
$$G = (\cos(u)+3)^2.$$

Now we just need to compute the Christoffel symbols:

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG-F^2)} \qquad \Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG-F^2)}$$
$$\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG-F^2)} \qquad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG-F^2)}$$
$$\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG-F^2)} \qquad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}.$$

So $E_u = E_v = F = F_u = F_v = G_v = 0$, and $G_u = -2(\cos(u)+3)\sin(u)$. Thus

$$\Gamma^1_{11} = 0 \qquad \Gamma^2_{11} = 0$$
$$\Gamma^1_{12} = 0 \qquad \Gamma^2_{12} = \frac{-\sin(u)}{\cos(u)+3}$$
$$\Gamma^1_{22} = -\sin(u)(\cos(u)+3) \qquad \Gamma^2_{22} = 0.$$

□

(2) Let $\boldsymbol{\sigma}\colon \mathbb{R}\times(0,2\pi)\to\mathbb{R}^3$ be the surface patch defined by

$$\boldsymbol{\sigma}(u,v) = \left(\cosh(u)\cos(v), \cosh(u)\sin(v), u\right).$$

Compute the Christoffel symbols Γ^k_{ij} for this patch.

Solution: This is a patch for the catenoid, which I computed the first fundamental form of in class (or see Pressley in section 9.1). At any rate, it has first fundamental form

$$E = \cosh^2(u)$$
$$F = 0$$
$$G = \cosh^2(u).$$

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Now we just need to compute the Christoffel symbols:

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)} \qquad \Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}$$
$$\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)} \qquad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)}$$
$$\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)} \qquad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.$$

So $E_v = G_v = F = F_u = F_v = 0$, and $E_u = G_u = 2\sinh(u)\cosh(u)$. Also, $EG - F^2 = \cosh^4(u)$. Thus

$$\Gamma^1_{11} = \tanh(u) \qquad \Gamma^2_{11} = 0$$
$$\Gamma^1_{12} = 0 \qquad \Gamma^2_{12} = \tanh(u)$$
$$\Gamma^1_{22} = -\tanh(u) \qquad \Gamma^2_{22} = 0.$$

□

(3) Compute the area of an n–sided polygon, whose sides are all geodesics, on the pseudosphere.

Solution: We'll use the corollary to Gauss-Bonnet that gives

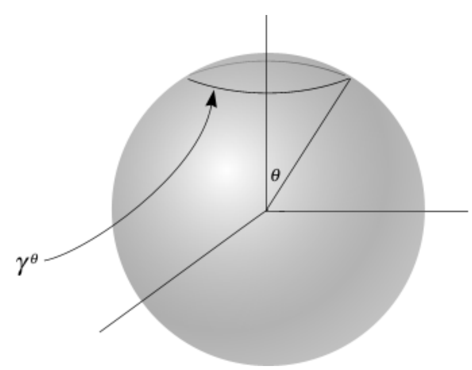
$$\sum_{i=1}^{n} \alpha_i = (n-2)\pi + \iint_{\text{int}(\boldsymbol{\gamma})} K d\mathcal{A},$$

where the α_i are the internal angles, and $\boldsymbol{\gamma}$ is the boundary curve. But $K = -1$ for the pseudosphere, so the integral is simply the negative of the area. Hence

$$\mathcal{A} = (n-2)\pi - \sum_{i=1}^{n} \alpha_i.$$

□

(4) For a fixed $\theta \in (0, \pi)$, let $\boldsymbol{\gamma}^\theta$ be a positively oriented unit-speed parameterization of the circle on S^2 with fixed θ (see figure).



Compute

$$\int_0^{\ell(\boldsymbol{\gamma}^\theta)} \kappa_g \, ds,$$

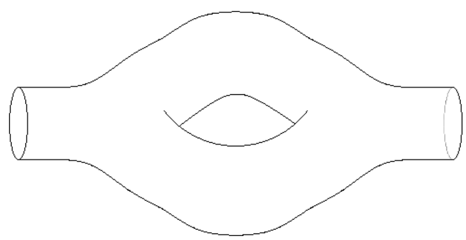
i.e. the integral of the geodesic curvature of $\boldsymbol{\gamma}^\theta$ around one period, as a function of θ.

Solution: Note I was a little tricky - using standard spherical angle coordinates, this is *not* a simple closed curve in a patch. There are a couple of ways to do this; one is to simply find a formula for the geodesic curvature of this curve (see the quiz) and do the integral. Another is to note that they are all simple closed curves in either chart of the stereographic projection atlas (for simplicity let's say the one that sends the south pole to infinity), so we can use Gauss-Bonnet. So all we need to note that the interior of all of these circles is the upper of the two regions bounded by the circle, and compute that area as a function of θ. Note we can compute its area via Archimedes's Theorem as $2\pi(1-\cos(\theta))$, since it gets mapped to a cylinder of radius 1 and height $(1-\cos(\theta))$ using the map from Archimedes's Theorem. So

$$\begin{aligned}\int_0^{\ell(\gamma^\theta)} \kappa_g \, ds &= 2\pi - (2\pi(1-\cos(\theta))) \\ &= 2\pi\cos(\theta).\end{aligned}$$

□

(5) A compact "surface with boundary" is the result of removing a finite number of open disjoint disks from a compact surface; for example, the figure below is the result of deleting two disjoint open disks from a torus.



You may assume (the true fact) that the Euler characteristic is well defined for surfaces with boundary. Compute the Euler characteristic of the pictured surface with boundary. If S is a compact surface with boundary with g holes and n disjoint circles for a boundary, guess a formula for the Euler characteristic of S.

Solution: Let $B_{g,n}$ be a compact surface of genus g minus the interior of n disjoint disks. Triangulate $B_{g,n}$ so that the boundary circles consist of edges. Then by adding n disks back as faces, we'll have a triangulation of the genus g surface T_g, which has Euler characteristic $2-2g$. So

$$\begin{aligned}\chi(B_{g,n}) &= \chi(T_g) - n \\ &= 2 - 2g - n.\end{aligned}$$

□