MATH 436
HOMEWORK 2
DUE SEPTEMBER 18, 2007

SOLUTIONS

(1) Let $\boldsymbol{\alpha}\colon (0,\infty) \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\alpha}(t) = \left(t^2, e^t, \sqrt{t}\right).$$

Compute the curvature of $\boldsymbol{\alpha}$.

Solution:

$$\begin{aligned}
\dot{\boldsymbol{\alpha}} &= \left(2t, e^t, \frac{1}{2\sqrt{t}}\right) \\
\ddot{\boldsymbol{\alpha}} &= \left(2, e^t, -\frac{1}{4}t^{-\frac{3}{2}}\right) \\
\|\ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}}\| &= \sqrt{\frac{e^{2t}\left(2\,t+1\right)^2}{16t^3} + \frac{9}{4t} + \left(2\,e^t - 2\,e^t t\right)^2} \\
\kappa(t) &= \frac{\|\ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}}\|}{\|\dot{\boldsymbol{\alpha}}\|^3} \\
&= \frac{\sqrt{\frac{e^{2t}(2\,t+1)^2}{16t^3} + \frac{9}{4t} + \left(2\,e^t - 2\,e^t t\right)^2}}{\left(\sqrt{4t^2 + e^{2t} + \frac{1}{4t}}\right)^3}
\end{aligned}$$

□

Date: September 11, 2007.

(2) Suppose $\boldsymbol{\beta}\colon \mathbb{R} \to \mathbb{R}^2$ is a (smooth) curve such that

$$\lim_{t\to\infty} \|\boldsymbol{\beta}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \|\boldsymbol{\beta}(t)\| = \infty.$$

Show that there is a $t_0 \in \mathbb{R}$ with $\|\boldsymbol{\beta}(t_0)\| \le \|\boldsymbol{\beta}(t)\|$ for all $t \in \mathbb{R}$.

Proof. I'll do this proof a couple of ways. The first will only require calculus, but is a little longer.

We know that $\|\boldsymbol{\beta}(t)\|$ is a continuous function $\mathbb{R} \to (0, \infty)$, since it is a composition of continuous functions. Let $b(t) = \|\boldsymbol{\beta}(t)\|$. Then

$$\lim_{t\to-\infty} b(t) = \infty \qquad \text{and} \qquad \lim_{t\to\infty} b(t) = \infty.$$

The first limit means for any $K_1 > 0$, there exits an $M_1 > 0$ such that if $t < -M_1$, then $b(t) > K_1$, and the second limit means that for any $K_2 > 0$, there exits an $M_2 > 0$ such that if $t > M_2$, then $b(t) > K_2$. So let's fix a value in the image of b for a reference, say $b(0)$, which we can set equal to both K_1 and K_2, and let M be the larger of M_1 and M_2. We then know that $b(t) > b(0)$ for all $t < -M$ and $t > M$ (that is, for all $t \notin [-M, M]$). So we know that b has a minimizer in $\mathbb{R}$ (that is, a $t_0 \in \mathbb{R}$ such that $b(t_0) \le b(t)$ for all $t \in \mathbb{R}$) if and only if b has a minimizer in $[-M, M]$, because $b(0)$ is strictly smaller than the image of anything outside of $[-M, M]$. But b is continuous on $[-M, M]$, so by the Extreme Value Theorem, b has a minimizer on $[-M, M]$, which must be a minimizer for all of $\mathbb{R}$.

If you've taken Math 432 (Topology), you might like the following proof: define b as above; the condition on $\boldsymbol{\beta}$ implies that b is a proper map $\mathbb{R} \to [0, \infty)$, i.e. the inverse image of a compact set is compact. So b extends to a map of the one-point compactifications; that is, b extends to a continuous map $\widetilde{b}\colon S^1 \to [0, \infty]$ (where we topologize $[0, \infty]$ to be homeomorphic to $[0, 1]$). Since S^1 is compact, so is its image under $\widetilde{b}$, which means that $\widetilde{b}$ has a minimizer (which isn't ∞), hence b has a minimizer. □

(3) Let $k\colon \mathbb{R} \to \mathbb{R}$ be the smooth function defined by

$$k(s) = s^2 \ln(s^2 + 1).$$

Plot a curve $\boldsymbol{\gamma}$ in $\mathbb{R}^2$ that has signed curvature equal to k. (You should use Maple, Mathematica, or some other computer program to produce the plot on some restricted domain; say $s = -10$ to $s = 10$.)

Solution: I used Maple with the following input:

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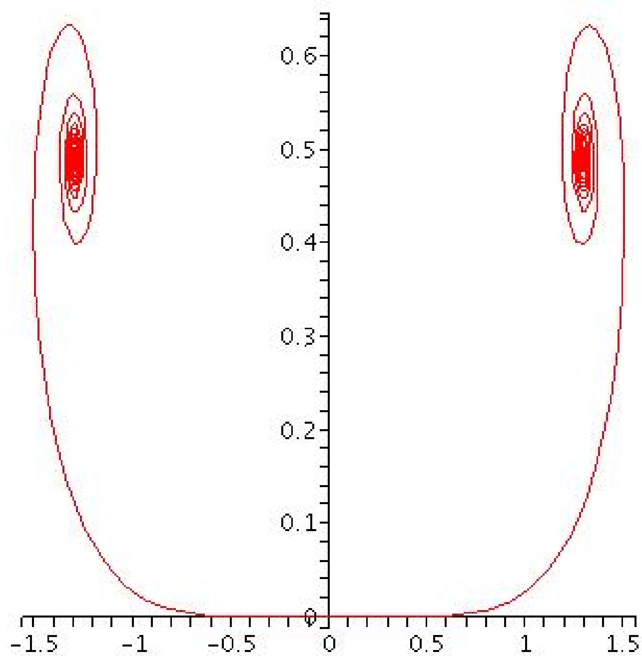
g:=x->x^2*ln(x^2+1);
f:=x->int(g(s),s=0..x);
plot([int(cos(f(x)),x=0..t),int(sin(f(x)),x=0..t),t=-10..10]);
```

This produced the following image:



□

(4) Let $\boldsymbol{\delta}\colon \mathbb{R} \to \mathbb{R}^2$ be defined by

$$\boldsymbol{\delta}(t) = (t\sin(t), t\cos(t)) \,.$$

Compute the curvature $\kappa(t)$ of $\boldsymbol{\delta}$. Since this curvature is always nonzero, we can say that for the signed curvature $\kappa_s(t)$ is either equal to $\kappa(t)$ for all t, or is equal to $-\kappa(t)$ for all t. Which is it? Explain how you know.

Solution:

$$\begin{aligned}
\dot{\boldsymbol{\delta}} &= (t\cos(t) + \sin(t), -t\sin(t) + \cos(t)) \\
\ddot{\boldsymbol{\delta}} &= (-t\sin(t) + 2\cos(t), -t\cos(t) - 2\sin(t)) \\
\ddot{\boldsymbol{\delta}} \times \dot{\boldsymbol{\delta}} &= \left(0, 0, t^2 + 2\right) \\
\kappa &= \frac{t^2+2}{\left(\sqrt{t^2+1}\right)^3}
\end{aligned}$$

Now note that reparametrizing by arc-length will only scale the velocity vector at each point, it will not change the direction. So if we're measuring the angle φ between the tangent vector and the x–axis, the *sign* of $\frac{d\varphi}{dt}$ will be the same as if we had parametrized by arc-length. Or, symbolically, since $\frac{ds}{dt} = \|\dot{\boldsymbol{\delta}}\| > 0$, by the chain rule $\frac{d\varphi}{dt}$ has the same sign as $\frac{d\varphi}{ds}$. In this case, that sign is negative, so

$$\kappa_s = -\kappa.$$

□

(5) Let $\boldsymbol{\varepsilon}\colon (a,b) \to \mathbb{R}^2$ be a unit-speed curve with signed curvature $\kappa_s(s) \neq 0$ for each $s \in (a,b)$, and let $\boldsymbol{n_s}(s)$ denote the signed normal vector. Let $\boldsymbol{\zeta}\colon (a,b) \to \mathbb{R}^2$ be the curve

$$\boldsymbol{\zeta}(s) = \boldsymbol{\varepsilon}(s) + \frac{1}{\kappa_s(s)}\boldsymbol{n_s}(s).$$

The curve $\boldsymbol{\zeta}$ is known as the *evolute* of $\boldsymbol{\varepsilon}$.

(a) Show for any $s \in (a,b)$, $\dot{\boldsymbol{\zeta}}(s)$ is perpendicular to $\dot{\boldsymbol{\varepsilon}}(s)$.

Proof. First, note that $\dot{\boldsymbol{n}}_{\boldsymbol{s}} = -\kappa_s \dot{\boldsymbol{\varepsilon}}$. This follows from differentiating the equation $\boldsymbol{n_s} \cdot \dot{\boldsymbol{\varepsilon}} = 0$. So we compute

$$\begin{aligned} \dot{\boldsymbol{\zeta}} &= \dot{\boldsymbol{\varepsilon}} + \frac{\kappa_s \dot{\boldsymbol{n}}_{\boldsymbol{s}} - \dot{\kappa}_s \boldsymbol{n_s}}{\left[\kappa_s\right]^2} \\ &= \dot{\boldsymbol{\varepsilon}} + \frac{\kappa_s\left(-\kappa_s \dot{\boldsymbol{\varepsilon}}\right) - \dot{\kappa}_s \boldsymbol{n_s}}{\left[\kappa_s\right]^2} \\ &= -\frac{\dot{\kappa}_s}{\left[\kappa_s\right]^2}\boldsymbol{n_s} \end{aligned}$$

This is a scalar multiple of $\boldsymbol{n_s}$, hence is perpendicular to $\dot{\boldsymbol{\varepsilon}}$ (I'll consider 0 to be perpendicular as well for simplicity). □

(b) Give an example of a regular curve whose evolute is not regular.

Solution: An easy way to do this is to note that since

$$\dot{\boldsymbol{\zeta}} = -\frac{\dot{\kappa}_s}{\left[\kappa_s\right]^2}\boldsymbol{n_s},$$

we need to have $\dot{\kappa}_s = 0$ for some point. Or for all points, if we just use a circle (which has constant curvature). □