MATH 436
HOMEWORK 2
DUE SEPTEMBER 18, 2007

(1) Let $\boldsymbol{\alpha}\colon (0,\infty) \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\alpha}(t) = \left(t^2, e^t, \sqrt{t}\right).$$

Compute the curvature of $\boldsymbol{\alpha}$.

(2) Suppose $\boldsymbol{\beta}\colon \mathbb{R} \to \mathbb{R}^2$ is a (smooth) curve such that

$$\lim_{t\to\infty} \|\boldsymbol{\beta}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \|\boldsymbol{\beta}(t)\| = \infty.$$

Show that there is a $t_0 \in \mathbb{R}$ with $\|\boldsymbol{\beta}(t_0)\| \leq \|\boldsymbol{\beta}(t)\|$ for all $t \in \mathbb{R}$.

(3) Let $k\colon \mathbb{R} \to \mathbb{R}$ be the smooth function defined by

$$k(s) = s^2 \ln(s^2 + 1).$$

Plot a curve $\boldsymbol{\gamma}$ in $\mathbb{R}^2$ that has signed curvature equal to k. (You should use Maple, Mathematica, or some other computer program to produce the plot on some restricted domain; say $s = -10$ to $s = 10$.)

(4) Let $\boldsymbol{\delta}\colon \mathbb{R} \to \mathbb{R}^2$ be defined by

$$\boldsymbol{\delta}(t) = (t\sin(t), t\cos(t)).$$

Compute the curvature $\kappa(t)$ of $\boldsymbol{\delta}$. Since this curvature is always nonzero, we can say that for the signed curvature $\kappa_s(t)$ is either equal to $\kappa(t)$ for all t, or is equal to $-\kappa(t)$ for all t. Which is it? Explain how you know.

(5) Let $\boldsymbol{\varepsilon}\colon (a,b) \to \mathbb{R}^2$ be a unit-speed curve with signed curvature $\kappa_s(s) \neq 0$ for each $s \in (a,b)$, and let $\boldsymbol{n_s}(s)$ denote the signed normal vector. Let $\boldsymbol{\zeta}\colon (a,b) \to \mathbb{R}^2$ be the curve

$$\boldsymbol{\zeta}(s) = \boldsymbol{\varepsilon}(s) + \frac{1}{\kappa_s(s)}\boldsymbol{n_s}(s).$$

The curve $\boldsymbol{\zeta}$ is known as the *evolute* of $\boldsymbol{\varepsilon}$.

(a) Show for any $s \in (a,b)$, $\dot{\boldsymbol{\zeta}}(s)$ is perpendicular to $\dot{\boldsymbol{\varepsilon}}(s)$.
(b) Give an example of a regular curve whose evolute is not regular.

Date: September 11, 2007.