MATH 436
HOMEWORK 3
DUE SEPTEMBER 25, 2007

SOLUTIONS

(1) Compute the curvature and torsion of the curve $\boldsymbol{\alpha}\colon \mathbb{R} \to \mathbb{R}^3$ given by

$$\boldsymbol{\alpha}(t) = (a\cosh(t), a\sinh(t), bt),$$

where $a, b \in \mathbb{R}$ are constants.

Solution: We'll use the following formulae:

$$\kappa = \frac{\| \ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}} \|}{\| \dot{\boldsymbol{\alpha}} \|^3} \tag{1}$$

$$\tau = \frac{(\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}) \cdot \dddot{\boldsymbol{\alpha}}}{\| \dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} \|^2}, \tag{2}$$

so we need to compute some derivatives:

$$\begin{aligned}
\boldsymbol{\alpha} &= (a\cosh(t), a\sinh(t), bt) \\
\dot{\boldsymbol{\alpha}} &= (a\sinh(t), a\cosh(t), b) \\
\ddot{\boldsymbol{\alpha}} &= (a\cosh(t), a\sinh(t), 0) \\
\dddot{\boldsymbol{\alpha}} &= (a\sinh(t), a\cosh(t), 0)\,.
\end{aligned}$$

So let's compute the things we need for κ and τ:

$$\begin{aligned}
\|\dot{\boldsymbol{\alpha}}\| &= \sqrt{a^2\sinh^2(t) + a^2\cosh^2(t) + b^2} \\
&= \sqrt{2a^2\sinh^2(t) + a^2 + b^2} \\
\ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}} &= \big(ab\sinh(t), -ab\cosh(t), a^2\big) \\
\| \ddot{\boldsymbol{\alpha}} \times \dot{\boldsymbol{\alpha}} \| &= \sqrt{2a^2b^2\sinh^2(t) + a^2(b^2 + a^2)} \\
\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}} &= \big(-ab\sinh(t), ab\cosh(t), -a^2\big) \\
(\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}) \cdot \dddot{\boldsymbol{\alpha}} &= a^2 b.
\end{aligned}$$

Plugging these into (1) and (2), we get

$$\begin{aligned}
\kappa &= \frac{\sqrt{2a^2b^2\sinh^2(t) + a^2(b^2 + a^2)}}{\big(2a^2\sinh^2(t) + a^2 + b^2\big)^{3/2}} \\
\tau &= \frac{a^2 b}{2a^2b^2\sinh^2(t) + a^2(b^2 + a^2)}
\end{aligned}$$

□

There are many, many different ways to write these, so your answer may look slightly different.

Date: September 18, 2007.

(2) Let $\boldsymbol{\beta}\colon (-1,1) \to \mathbb{R}^3$ be defined by

$$\boldsymbol{\beta}(s) = \left(\frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}} \right).$$

Show $\boldsymbol{\beta}$ is a unit speed curve. Compute its tangent vector, principal normal vector, and binormal vector, and compute each of their first derivatives with respect to s.

Proof. We compute $\dot{\boldsymbol{\beta}}$ to verify that we have a unit-speed curve:

$$\begin{aligned} \dot{\boldsymbol{\beta}} &= \left(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}} \right) \\ \left\| \dot{\boldsymbol{\beta}} \right\|^2 &= \frac{1+s}{4} + \frac{1-s}{4} + \frac{1}{2} \\ &= 1. \end{aligned}$$

So we have the tangent vector $\dot{\boldsymbol{\beta}}$, we now need to compute $\ddot{\boldsymbol{\beta}}$ in order to compute the principal normal.

$$\begin{aligned} \ddot{\boldsymbol{\beta}} &= \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0 \right) \\ \left\| \ddot{\boldsymbol{\beta}} \right\|^2 &= \frac{1}{16(1+s)} + \frac{1}{16(1-s)} \\ &= \frac{1}{8(1-s^2)}. \end{aligned}$$

So $\kappa = \frac{1}{\sqrt{8(1-s^2)}}$, and the principal normal is

$$\begin{aligned} \boldsymbol{n} &= \frac{1}{\kappa} \ddot{\boldsymbol{\beta}} \\ &= \left(\sqrt{\frac{1-s}{2}}, \sqrt{\frac{1+s}{2}}, 0 \right). \end{aligned}$$

And it is now trivial to compute the binormal:

$$\begin{aligned} \boldsymbol{b} &= \dot{\boldsymbol{\beta}} \times \boldsymbol{n} \\ &= \left(-\frac{\sqrt{1+s}}{2}, \frac{\sqrt{1-s}}{2}, \frac{\sqrt{1+s}}{2}\sqrt{\frac{1+s}{2}} + \frac{\sqrt{1-s}}{2}\sqrt{\frac{1-s}{2}} \right) \\ &= \left(-\frac{\sqrt{1+s}}{2}, \frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}} \right). \end{aligned}$$

So we've got the tangent vector and its derivative, the principal normal, and the binormal; finding $\dot{\boldsymbol{b}}$ gives us τ, which we could use to compute the derivative of the principal normal, but it is easier to compute directly.

$$\begin{aligned} \dot{\boldsymbol{b}} &= \left(-\frac{1}{4\sqrt{1+s}}, -\frac{1}{4\sqrt{1-s}}, 0 \right) \\ &= -\ddot{\boldsymbol{\beta}} \\ &= -\kappa \boldsymbol{n}. \end{aligned}$$

This means $\tau = \kappa = \frac{1}{\sqrt{8(1-s^2)}}$. We now can compute

$$\dot{\boldsymbol{n}} = \left(-\frac{1}{\sqrt{8(1-s)}}, \frac{1}{\sqrt{8(1+s)}}, 0 \right).$$

□

(3) Is there a simple closed curve in the plane with length equal to 6 feet bounding an area of 3 square feet? If so, give an example; if not, tell why not.

Answer: No, this would violate the Isoperimetric Inequality, since $3 > \frac{36}{4\pi}$. □

(4) Let $\boldsymbol{\gamma}\colon \mathbb{R} \to \mathbb{R}^2$ be the curve

$$\boldsymbol{\gamma}(t) = \left([\cos(3t) + 2] \cos(t), [\cos(3t) + 2] \sin(t) \right).$$

Show $\boldsymbol{\gamma}$ is a simple closed curve, and compute the area of its interior.

Proof. It is clear that γ has period 2π. Computing the area of the interior is pretty easy; let's identify some functions we'll need.

$$\begin{aligned}
x(t) &= [\cos(3t) + 2] \cos(t) \\
&= \cos(t) \cos(3t) + 2 \cos(t) \\
y(t) &= [\cos(3t) + 2] \sin(t) \\
&= \sin(t) \cos(3t) + 2 \sin(t) \\
\dot{x}(t) &= -\sin(t) \cos(3t) - 3 \cos(t) \sin(3t) - 2 \sin(t) \\
\dot{y}(t) &= \cos(t) \cos(3t) - 3 \sin(t) \sin(3t) + 2 \cos(t).
\end{aligned}$$

Now the area of the interior is

$$\begin{aligned}
\mathcal{A}\left(\operatorname{int}(\boldsymbol{\gamma})\right) &= \frac{1}{2} \int_0^{2\pi} \left(x\dot{y} - y\dot{x} \right) \, dt \\
&= \frac{9\pi}{2}.
\end{aligned}$$

I used Maple to evaluate this integral, because I am lazy. □