MATH 436
HOMEWORK 4
DUE OCTOBER 9, 2007

SOLUTIONS

(1) Recall a set $U \subset \mathbb{R}^n$ is called *open* if for each $x \in U$, there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$, where

$$B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|x - y\| < \varepsilon\}.$$

(a) Show that $\mathbb{R}^n$ and the empty set $\varnothing$ are open sets.
(b) If A is any indexing set, and for each $\alpha \in A$, U_α is an open subset of $\mathbb{R}^n$, show that $\bigcup_{\alpha \in A} U_\alpha$ is an open subset of $\mathbb{R}^n$.
(c) If $U_1, U_2, \ldots, U_n$ is a (finite) collection of open subsets of $\mathbb{R}^n$, show that $\bigcap_{i=1}^n U_i$ is an open subset of $\mathbb{R}^n$.
(d) Give an infinite collection of open subsets of $\mathbb{R}$ whose intersection is *not* open.

Proof. We'll just go down the list.
(a) The empty set $\varnothing$ is open vacuously, we don't have to do anything. $\mathbb{R}^n$ is open since by definition, all ε–balls are contained in $\mathbb{R}^n$.
(b) Let $x \in \bigcup_{\alpha \in A} U_\alpha$. Then $x \in U_\alpha$ for some $\alpha \in A$, so there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U_\alpha$. But $U_\alpha \subset \bigcup_{\alpha \in A} U_\alpha$, so

$$B_\varepsilon(x) \subset \bigcup_{\alpha \in A} U_\alpha.$$

(c) Let $x \in \bigcap_{i=1}^n U_i$, then $x \in U_i$ for each i. So for each i, there is an $\varepsilon_i > 0$ such that $B_{\varepsilon_i}(x) \subset U_i$. Let $\varepsilon = \min_i(\varepsilon_i)$, i.e. ε is the smallest of all of the ε_i's, so $\varepsilon \leq \varepsilon_i$ for each i. This means $B_\varepsilon(x) \subset B_{\varepsilon_i}(x)$ for each i, so $B_\varepsilon(x) \subset U_i$ for each i, hence

$$B_\varepsilon(x) \subset \bigcap_{i=1}^n U_i.$$

(d) The collection of open intervals

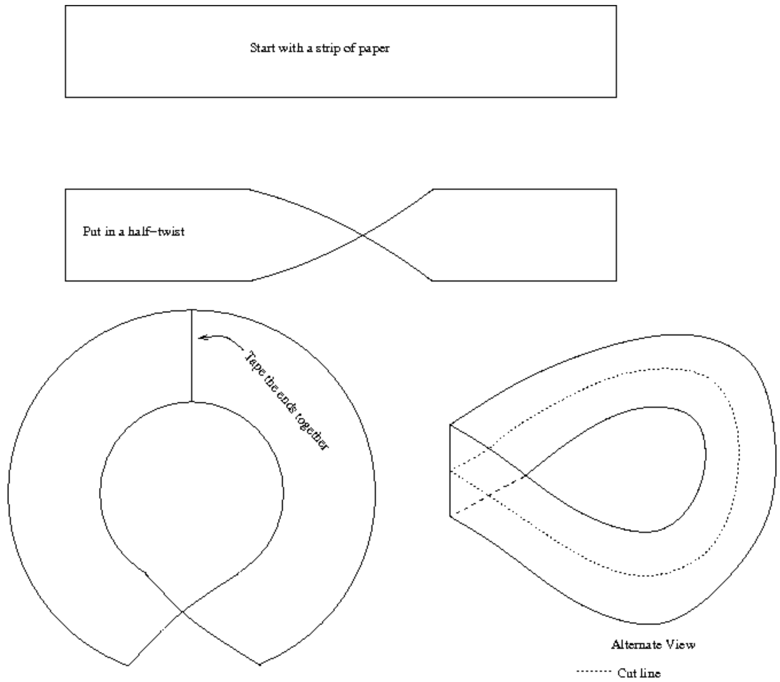
$$U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$$

has intersection $\{0\}$, which isn't open.

□

Date: October 2, 2007.

(2) Using your Möbius band from class, or using one you make using the diagram below, cut a Möbius band down the middle along the length of the band (see diagram below with dotted line showing where to cut). Is the resulting surface orientable? Why or why not?



Solution: This is orientable. If you start anyplace on the surface and decide on a normal direction, you can go all the way around with a consistent choice of normal. □

(3) As usual, let S^2 denote the unit sphere in $\mathbb{R}^3$,

$$S^2 = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2+y^2+z^2 = 1\right\},$$

and let $NP = (0,0,1)$ and $SP = (0,0,-1)$, the north and south poles respectively. In class we got a surface patch $\boldsymbol{\sigma}_{NP} \colon \mathbb{R}^2 \to S^2 - \{NP\}$ by

$$\boldsymbol{\sigma}_{NP}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right),$$

and it was left as an exercise to find $\boldsymbol{\sigma}_{SP} \colon \mathbb{R}^2 \to S^2 - \{SP\}$. You should have obtained the following (you do not have to redo it now):

$$\boldsymbol{\sigma}_{SP}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}\right).$$

Compute the transition maps for this atlas (there should be two). You'll need the inverses of the surface patches.

Solution: We can easily compute the inverses:

$$\boldsymbol{\sigma}_{NP}^{-1} = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

$$\boldsymbol{\sigma}_{SP}^{-1} = \left(\frac{x}{1+z}, \frac{y}{1+z}\right).$$

Now we compute the compositions:

$$\begin{aligned}
\left(\boldsymbol{\sigma}_{SP}^{-1} \circ \boldsymbol{\sigma}_{NP}\right)(u,v) &= \left(\frac{\frac{2u}{u^2+v^2+1}}{1+\frac{u^2+v^2-1}{u^2+v^2+1}}, \frac{\frac{2v}{u^2+v^2+1}}{1+\frac{u^2+v^2-1}{u^2+v^2+1}}\right) \\
&= \left(\frac{\frac{2u}{u^2+v^2+1}}{\frac{u^2+v^2+1}{u^2+v^2+1}+\frac{u^2+v^2-1}{u^2+v^2+1}}, \frac{\frac{2v}{u^2+v^2+1}}{\frac{u^2+v^2+1}{u^2+v^2+1}+\frac{u^2+v^2-1}{u^2+v^2+1}}\right) \\
&= \left(\frac{\frac{2u}{u^2+v^2+1}}{\frac{2u^2+2v^2}{u^2+v^2+1}}, \frac{\frac{2v}{u^2+v^2+1}}{\frac{2u^2+2v^2}{u^2+v^2+1}}\right) \\
&= \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right), \\
\left(\boldsymbol{\sigma}_{NP}^{-1} \circ \boldsymbol{\sigma}_{SP}\right)(u,v) &= \left(\frac{\frac{2u}{u^2+v^2+1}}{1-\frac{1-u^2-v^2}{u^2+v^2+1}}, \frac{\frac{2v}{u^2+v^2+1}}{1-\frac{1-u^2-v^2}{u^2+v^2+1}}\right) \\
&= \left(\frac{\frac{2u}{u^2+v^2+1}}{\frac{u^2+v^2+1}{u^2+v^2+1}-\frac{1-u^2-v^2}{u^2+v^2+1}}, \frac{\frac{2v}{u^2+v^2+1}}{\frac{u^2+v^2+1}{u^2+v^2+1}-\frac{1-u^2-v^2}{u^2+v^2+1}}\right) \\
&= \left(\frac{\frac{2u}{u^2+v^2+1}}{\frac{2u^2+2v^2}{u^2+v^2+1}}, \frac{\frac{2v}{u^2+v^2+1}}{\frac{2u^2+2v^2}{u^2+v^2+1}}\right) \\
&= \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right).
\end{aligned}$$

□

(4) In Exercise 4.2 in the text, Pressley defines the circular cylinder

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 = 1\right\}.$$

(See the solution in the back of the book for a surface patch).

(a) Show that S is a smooth surface.

Proof. The surface patch given in Pressley is $\boldsymbol{\sigma}\colon U \to \mathbb{R}^3$, where

$$U = \left\{(u, v) \in \mathbb{R}^2 \;\middle|\; 0 < u^2 + v^2 < \pi^2\right\},$$

given by

$$\boldsymbol{\sigma}(u, v) = \left(\frac{u}{\sqrt{u^2 + v^2}}, \frac{v}{\sqrt{u^2 + v^2}}, \tan\left(\sqrt{u^2 + v^2} - \frac{\pi}{2}\right)\right).$$

It is a routine exercise to compute

$$\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = \left(-\frac{u\left(1 + \cot^2\left(\sqrt{u^2 + v^2}\right)\right)}{u^2 + v^2}, -\frac{v\left(1 + \cot^2\left(\sqrt{u^2 + v^2}\right)\right)}{u^2 + v^2}, 0\right).$$

But u and v can't both be equal to zero in U, and also $\cot^2\left(\sqrt{u^2 + v^2}\right)$ is always positive in U, so $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v$ is always nonzero in U, hence $\boldsymbol{\sigma}$ is a regular patch. So we have an atlas consisting of a single regular patch, hence the surface is a smooth surface. □

(b) Let $f\colon S \to S^2$ be the map

$$f(x, y, z) = (x, y, 0).$$

Show that f is a smooth map.

Proof. Since we have the inverses for stereographic projection above, I'll use those. We want to compute $\boldsymbol{\sigma}_{NP}^{-1} \circ f \circ \boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_{SP}^{-1} \circ f \circ \boldsymbol{\sigma}$. Since the image of f is completely contained in either of the two patches, and the transition maps are smooth, it will suffice to show that $\boldsymbol{\sigma}_{NP}^{-1} \circ f \circ \boldsymbol{\sigma}$ is smooth, since we can get $\boldsymbol{\sigma}_{SP}^{-1} \circ f \circ \boldsymbol{\sigma}$ from that by composition with a transition map.

$$\boldsymbol{\sigma}_{NP}^{-1} \circ f \circ \boldsymbol{\sigma}(u, v) = \boldsymbol{\sigma}_{NP}^{-1} \circ (f \circ \boldsymbol{\sigma})(u, v)$$

$$(f \circ \boldsymbol{\sigma})(u, v) = \left(\frac{u}{\sqrt{u^2 + v^2}}, \frac{v}{\sqrt{u^2 + v^2}}, 0\right)$$

$$\boldsymbol{\sigma}_{NP}^{-1} \circ f \circ \boldsymbol{\sigma}(u, v) = \left(\frac{u}{\sqrt{u^2 + v^2}}, \frac{v}{\sqrt{u^2 + v^2}}\right).$$

This is smooth on U since $u^2 + v^2 > 0$ on U. □

(5) Let $\boldsymbol{\gamma}\colon \mathbb{R} \to \mathbb{R}^3$ be the curve

$$\boldsymbol{\gamma}(t) = (\cos(t), \sin(t), t)\,.$$

Show that the image of $\boldsymbol{\gamma}$ is contained in the circular cylinder from the previous problem, and give functions $u, v\colon \mathbb{R} \to \mathbb{R}$ such that

$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}\left(u(t), v(t)\right),$$

where $\boldsymbol{\sigma}$ is the single surface patch for the cylinder given in the solution to Exercise 4.2 in Pressley. Plot $(u(t), v(t))$ (**not** $\boldsymbol{\sigma}(u(t), v(t))$ - the plot should be in the plane) for an appropriate range of values of t (say $t = -50$ to $t = 50$).

Solution: To show that this curve is contained on the cylinder, we simply need to note that $\boldsymbol{\gamma}(t)$ satisfies the defining equation of the surface for all t, which is clear. To find the local parameterization $(u(t), v(t))$, we can first find $r(t)$, where

$$r(t) = \sqrt{u(t)^2 + v(t)^2}.$$

But if

$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}\left(u(t), v(t)\right),$$

then we know that

$$\tan\left(r(t) - \frac{\pi}{2}\right) = t.$$

Solving this, we see

$$r(t) = \frac{\pi}{2} + \arctan(t).$$

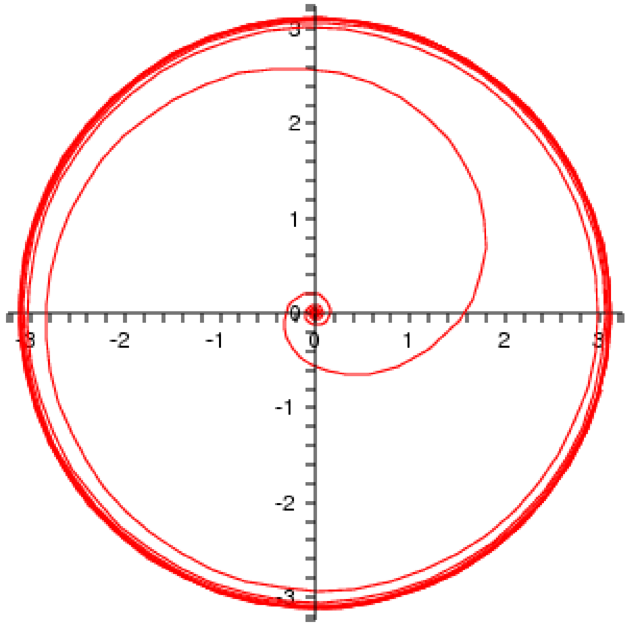
Then we must have

$$\frac{u(t)}{r(t)} = \cos(t), \quad \text{and}$$
$$\frac{v(t)}{r(t)} = \sin(t).$$

So we can now solve:

$$u(t) = \cos(t)\left(\frac{\pi}{2} + \arctan(t)\right),$$
$$v(t) = \sin(t)\left(\frac{\pi}{2} + \arctan(t)\right).$$

Here's a plot of $(u(t), v(t))$ for $-50 \leq t \leq 50$:



3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

□