MATH 436
HOMEWORK 4
DUE OCTOBER 9, 2007

(1) Recall a set $U \subset \mathbb{R}^n$ is called *open* if for each $x \in U$, there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$, where

$$B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|x - y\| < \varepsilon\}.$$

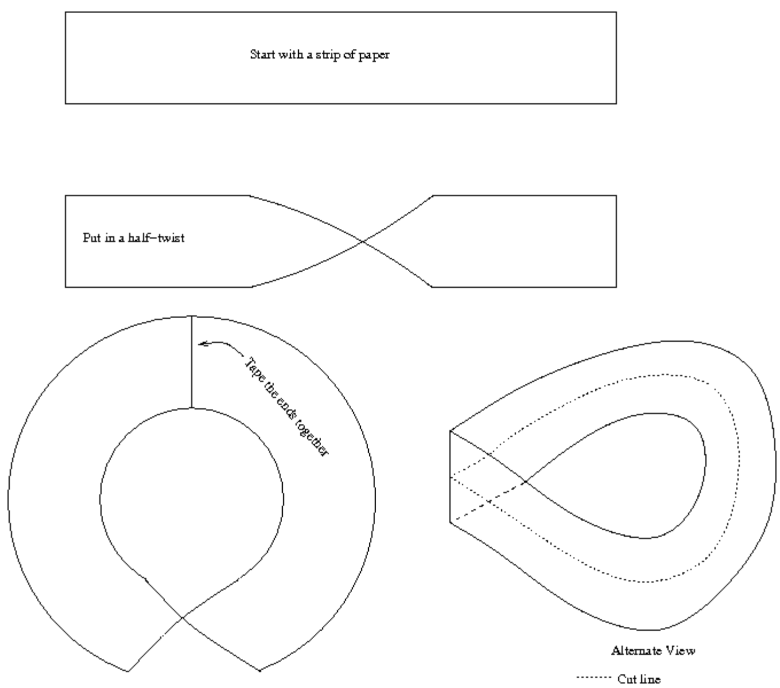
(a) Show that $\mathbb{R}^n$ and the empty set $\varnothing$ are open sets.

(b) If A is any indexing set, and for each $\alpha \in A$, U_α is an open subset of $\mathbb{R}^n$, show that $\bigcup_{\alpha \in A} U_\alpha$ is an open subset of $\mathbb{R}^n$.

(c) If $U_1, U_2, \ldots, U_n$ is a (finite) collection of open subsets of $\mathbb{R}^n$, show that $\bigcap_{i=1}^{n} U_i$ is an open subset of $\mathbb{R}^n$.

(d) Give an infinite collection of open subsets of $\mathbb{R}$ whose intersection is *not* open.

(2) Using your Möbius band from class, or using one you make using the diagram below, cut a Möbius band down the middle along the length of the band (see diagram below with dotted line showing where to cut). Is the resulting surface orientable? Why or why not?



Date: October 2, 2007.

(3) As usual, let S^2 denote the unit sphere in $\mathbb{R}^3$,
$$S^2 = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2+y^2+z^2=1\right\},$$
and let $NP = (0,0,1)$ and $SP = (0,0,-1)$, the north and south poles respectively. In class we got a surface patch $\boldsymbol{\sigma}_{NP}\colon \mathbb{R}^2 \to S^2 - \{NP\}$ by
$$\boldsymbol{\sigma}_{NP}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right),$$
and it was left as an exercise to find $\boldsymbol{\sigma}_{SP}\colon \mathbb{R}^2 \to S^2 - \{SP\}$. You should have obtained the following (you do not have to redo it now):
$$\boldsymbol{\sigma}_{SP}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}\right).$$
Compute the transition maps for this atlas (there should be two). You'll need the inverses of the surface patches.

(4) In Exercise 4.2 in the text, Pressley defines the circular cylinder
$$S = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2+y^2=1\right\}.$$
(See the solution in the back of the book for a surface patch).
 (a) Show that S is a smooth surface.
 (b) Let $f\colon S \to S^2$ be the map
$$f(x,y,z) = (x,y,0).$$
Show that f is a smooth map.

(5) Let $\boldsymbol{\gamma}\colon \mathbb{R} \to \mathbb{R}^3$ be the curve
$$\boldsymbol{\gamma}(t) = (\cos(t), \sin(t), t).$$
Show that the image of $\boldsymbol{\gamma}$ is contained in the circular cylinder from the previous problem, and give functions $u, v\colon \mathbb{R} \to \mathbb{R}$ such that
$$\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}\left(u(t), v(t)\right),$$
where $\boldsymbol{\sigma}$ is the single surface patch for the cylinder given in the solution to Exercise 4.2 in Pressley. Plot $(u(t), v(t))$ (**not** $\boldsymbol{\sigma}(u(t), v(t))$ - the plot should be in the plane) for an appropriate range of values of t (say $t = -50$ to $t = 50$).