MATH 436
HOMEWORK 5
DUE OCTOBER 16, 2007

SOLUTIONS

(1) Let $\boldsymbol{\sigma}\colon (0,2\pi)\times(0,2\pi)\to\mathbb{R}^3$ be the surface patch defined by

$$\boldsymbol{\sigma}(u,v) = ((\cos(u)+3)\cos(v), (\cos(u)+3)\sin(v), \sin(u)) .$$

Show that this surface patch is regular, and compute its first fundamental form. Describe this surface.

Solution: We simply compute

$$\boldsymbol{\sigma}_u\times\boldsymbol{\sigma}_v = (-\cos(u)(\cos(u)+3)\cos(v), -\cos(u)(\cos(u)+3)\sin(v), -\sin(u)(\cos(u)+3)) .$$

Since $(\cos(u)+3)$ is always nonzero, for this vector to be zero we will either need both $\cos(u)$ and $\sin(u)$ to be simultaneously zero, or $\cos(v)$ and $\sin(v)$ to be simultaneously zero, which is impossible.

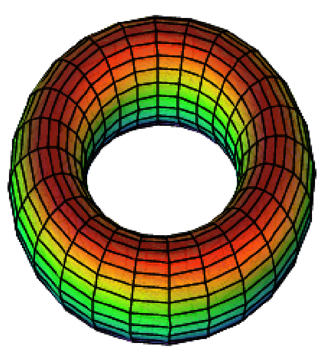
To compute the first fundamental form:

$$E = \boldsymbol{\sigma}_u\cdot\boldsymbol{\sigma}_u = 1$$
$$F = \boldsymbol{\sigma}_u\cdot\boldsymbol{\sigma}_v = 0$$
$$G = \boldsymbol{\sigma}_v\cdot\boldsymbol{\sigma}_v = (\cos(u)+3)^2$$

So the first fundamental form for $\boldsymbol{\sigma}$ is

$$ds^2 = du^2 + (\cos(u)+3)^2 dv^2.$$

Here's a plot of the surface, which is a torus:



□

Date: October 9, 2007.

(2) Let $U \subset \mathbb{R}^2$ be a (nonempty) open subset.

(a) Let $S_1 = \{(x, y, 0) \in \mathbb{R}^3 \mid (x, y) \in U\}$. Show that S_1 is a smooth surface in $\mathbb{R}^3$ by finding a regular surface patch $\boldsymbol{\sigma}_1$ (don't overthink this).

Proof. A simple regular patch is $\boldsymbol{\sigma}_1 \colon U \to \mathbb{R}^3$ given by:

$$\boldsymbol{\sigma}_1(u, v) = (u, v, 0).$$

This is clearly regular. □

(b) Let $\boldsymbol{\sigma}_2 \colon U \to \mathbb{R}^3$ be a regular surface patch, and let $S_2 = \boldsymbol{\sigma}_2(U)$; the image of $\boldsymbol{\sigma}_2$ in $\mathbb{R}^3$. Let $f \colon S_1 \to S_2$ be the map

$$f(x, y, 0) = \boldsymbol{\sigma}_2(x, y).$$

Show that f is smooth.

Proof. We simply need to compute $\boldsymbol{\sigma}_2^{-1} \circ f \circ \boldsymbol{\sigma}_1$, and show that it is smooth. But this composition is clearly the identity, which is a smooth map. It is also clear that f is a diffeomorphism. □

(c) Under what conditions on $\boldsymbol{\sigma}_2$ is f an isometry?

Solution: The diffeomorphism f will be an isometry if and only if the first fundamental form of $\boldsymbol{\sigma}_2$ is $du^2 + dv^2$, since it is clear that $\boldsymbol{\sigma}_2 = f \circ \boldsymbol{\sigma}_1$. □

(3) A surface patch for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

can be given by

$$\boldsymbol{\sigma}(u, v) = (a\cos(u)\sin(v), b\sin(u)\sin(v), c\cos(v)) .$$

Compute the first fundamental form for this surface patch.

Solution: This is simply a computation.

$$E = \sin^2(v)\left((b^2 - a^2)\cos^2(u) + a^2\right)$$
$$F = \sin(u)\sin(v)\cos(u)\cos(v)(b^2 - a^2)$$
$$G = \cos^2(u)\cos^2(v)(a^2 - b^2) + \cos^2(v)(b^2 - c^2) + c^2.$$

So now

$$ds^2 = \sin^2(v)\left((b^2 - a^2)\cos^2(u) + a^2\right)du^2 + 2\sin(u)\sin(v)\cos(u)\cos(v)(b^2 - a^2)dudv + \cos^2(u)\cos^2(v)(a^2 - b^2) + (\cos^2(v)(b^2 - c^2) + c^2)dv^2.$$

You should eyeball this and notice that it is the usual first fundamental form for the unit sphere when $a = b = c = 1$. □

(4) Let $U \subset \mathbb{R}^2$ be an open set, and suppose $f\colon U \to \mathbb{R}$ is smooth. Define the *graph* of f, $\Gamma(f) \subset \mathbb{R}^3$, to be

$$\Gamma(f) = \{(x, y, f(x, y)) \mid (x, y) \in U\} .$$

Show $\Gamma(f)$ is a smooth surface, and compute its first fundamental form.

Solution: First we find a regular surface patch $\boldsymbol{\sigma}\colon U \to \mathbb{R}^3$; an easy one is

$$\boldsymbol{\sigma}(u, v) = (u, v, f(u, v)).$$

We can easily compute

$$\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (-f_u, -f_v, 1),$$

which is always nonzero. So $\Gamma(f)$ is a smooth surface since it can be covered by a single regular surface patch. We now compute

$$ds^2 = (1 + f_u^2)du^2 + 2f_u f_v dudv + (1 + f_v^2)dv^2.$$

□

(5) Let $f\colon \mathbb{R}^3 \to \mathbb{R}$, given in cylindrical coordinates, be given by

$$f(r,\varphi,z) = (r-3)^2 + z^2 - 1.$$

Show that the set $f^{-1}(0)$ is a smooth surface (this may require rectangular coordinates). Sketch this surface. *Hint:* First sketch in the rz–plane.

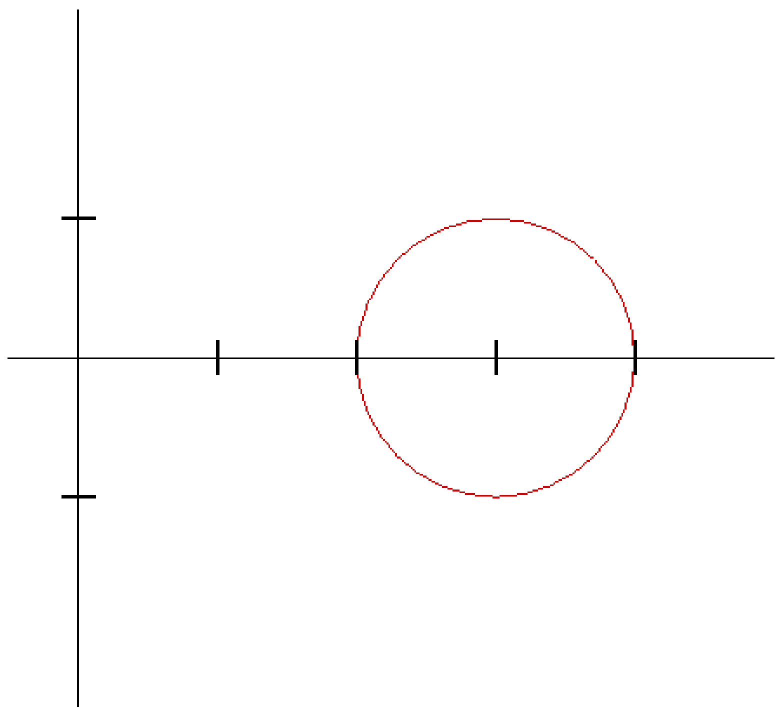
Solution: In rectangular coordinates, $f(x,y,z) = (\sqrt{x^2+y^2}-3)^2 + z^2 - 1$. So

$$f_x = 2x\,\frac{\left(\sqrt{x^2+y^2}-3\right)}{\sqrt{x^2+y^2}}$$

$$f_y = 2y\,\frac{\left(\sqrt{x^2+y^2}-3\right)}{\sqrt{x^2+y^2}}$$

$$f_z = 2z.$$

This is always nonzero; first note that x, y, z can't all be zero and satisfy $f(x,y,z) = 0$. Also, z and $r - 3 = \sqrt{x^2+y^2} - 3$ can't both be zero to satisfy $f(x,y,z) = 0$. So this is a regular surface. There is an image in problem 1; these are two different descriptions of the same surface. In the rz–plane, the plot is:



Now imagine rotating this circle around the z–axis to get the torus. □