MATH 436
HOMEWORK 5
DUE OCTOBER 16, 2007

(1) Let $\boldsymbol{\sigma}\colon (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3$ be the surface patch defined by

$$\boldsymbol{\sigma}(u, v) = ((\cos(u) + 3)\cos(v), (\cos(u) + 3)\sin(v), \sin(u)) .$$

Show that this surface patch is regular, and compute its first fundamental form. Describe this surface.

(2) Let $U \subset \mathbb{R}^2$ be a (nonempty) open subset.

(a) Let $S_1 = \{(x, y, 0) \in \mathbb{R}^3 \mid (x, y) \in U\}$. Show that S_1 is a smooth surface in $\mathbb{R}^3$ by finding a regular surface patch $\boldsymbol{\sigma}_1$ (don't overthink this).

(b) Let $\boldsymbol{\sigma}_2\colon U \to \mathbb{R}^3$ be a regular surface patch, and let $S_2 = \boldsymbol{\sigma}_2(U)$; the image of $\boldsymbol{\sigma}_2$ in $\mathbb{R}^3$. Let $f\colon S_1 \to S_2$ be the map

$$f(x, y, 0) = \boldsymbol{\sigma}_2(x, y).$$

Show that f is smooth.

(c) Under what conditions on $\boldsymbol{\sigma}_2$ is f an isometry?

(3) A surface patch for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

can be given by

$$\boldsymbol{\sigma}(u, v) = (a\cos(u)\sin(v), b\sin(u)\sin(v), c\cos(v)) .$$

Compute the first fundamental form for this surface patch.

(4) Let $U \subset \mathbb{R}^2$ be an open set, and suppose $f\colon U \to \mathbb{R}$ is smooth. Define the *graph* of f, $\Gamma(f) \subset \mathbb{R}^3$, to be

$$\Gamma(f) = \{(x, y, f(x, y)) \mid (x, y) \in U\} .$$

Show $\Gamma(f)$ is a smooth surface, and compute its first fundamental form.

(5) Let $f\colon \mathbb{R}^3 \to \mathbb{R}$, given in cylindrical coordinates, be given by

$$f(r, \varphi, z) = (r - 3)^2 + z^2 - 1.$$

Show that the set $f^{-1}(0)$ is a smooth surface (this may require rectangular coordinates). Sketch this surface. *Hint:* First sketch in the rz–plane.

Date: October 9, 2007.