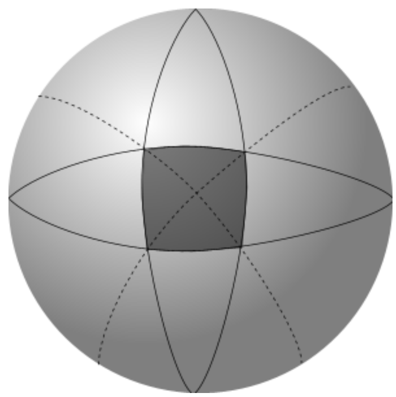
MATH 436
HOMEWORK 6
DUE OCTOBER 30, 2007

SOLUTIONS

(1) Suppose that Q is a quadrilateral on the unit sphere S^2 whose sides are all (pieces of) great circles. Suppose that each of the two great-circle arcs connecting opposite angles of Q are in the interior of Q. An example is pictured below, with Q shaded.



Find a formula for the area of Q in terms of the angles of Q. Generalize to an arbitrary polygon.

Solution: Note that each of the diagonals splits the quadrilateral into two triangles. Let's just consider one of the diagonals, and the two resulting triangles. Summing the angles of the two triangles gives the sum of the angles of the quadrilateral, and summing the area of the two triangles gives the area of the quadrilateral. So the area of the quadrilateral is the sum of its angles minus 2π. To generalize, we need to have a polygon where the great-circle arc connecting any one vertex to any of the other non-adjacent vertices is contained in the polygon (for example a convex polygon). In this case, we can split an n-gon into $n-2$ triangles, so the area of such an n-gon is the sum of the angles minus $(n-2)\pi$. □

(2) One possible smooth atlas for the unit cylinder $C = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 = 1\right\}$ follows: Let $U = (0, 2\pi) \times \mathbb{R} \subset \mathbb{R}^2$ and $U' = (-\pi, \pi) \times \mathbb{R} \subset \mathbb{R}^2$. Let $\boldsymbol{\sigma}\colon U \to \mathbb{R}^3$ be given by

$$\boldsymbol{\sigma}(u, v) = (\cos(u), \sin(u), v)\,,$$

and let $\boldsymbol{\sigma}'\colon U' \to \mathbb{R}^3$ be given by

$$\boldsymbol{\sigma}'(u, v) = (\cos(u), \sin(u), v)\,.$$

(It is intentional that they have the same formula.) Compute the second fundamental form for the patches in this atlas.

Solution: This is a straightforward computation.

$$\begin{aligned} E &= 1 \\ F &= 0 \\ G &= 1 \\ L &= -1 \\ M &= 0 \\ N &= 0 \end{aligned}$$

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So the first fundamental form is

$$du^2 + dv^2,$$

and the second fundamental form is

$$-du^2.$$

□

(3) Draw examples of curves with vanishing geodesic curvature on the following surfaces:
   (a) The sphere $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.
   (b) The torus $\left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, (\sqrt{x^2 + y^2} - 3)^2 + z^2 = 1\right\}$.
   (c) The cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.

Solution: Basically all I had in mind was that you'd draw several curves that are the intersection of planes that are normal to each surface with the surface. Things that spring to mind for me are:
   (a) Great circles. That's pretty much it.
   (b) Circles of constant φ (in cylindrical coordinates), or the circle that is the intersection of the xy–plane with the torus.
   (c) Lines of constant φ (in cylindrical coordinates), or circles of constant z.

□

(4) Let $U \subset \mathbb{R}^2$ and $f\colon U \to \mathbb{R}$ be a smooth map. Let

$$\Gamma(f) = \left\{(x, y, f(x, y)) \in \mathbb{R}^3 \,\middle|\, (x, y) \in U\right\}.$$

   (a) Compute the second fundamental form of (an appropriate regular surface patch of) $\Gamma(f)$.

   Solution: The obvious patch is

$$\boldsymbol{\sigma}(u, v) = (u, v, f(u, v)).$$

   So $\boldsymbol{\sigma}_u = (1, 0, f_u)$, $\boldsymbol{\sigma}_v = (0, 1, f_v)$, and the normal vector is $\boldsymbol{N} = \frac{1}{\sqrt{1+f_u^2+f_v^2}}(-f_u, -f_v, 1)$. Then $\boldsymbol{\sigma}_{uu} = (0, 0, f_{uu})$, $\boldsymbol{\sigma}_{uv} = (0, 0, f_{uv})$, and $\boldsymbol{\sigma}_{vv} = (0, 0, f_{vv})$. So $L = \frac{f_{uu}}{\sqrt{1+f_u^2+f_v^2}}$, $M = \frac{f_{uv}}{\sqrt{1+f_u^2+f_v^2}}$, and $N = \frac{f_{vv}}{\sqrt{1+f_u^2+f_v^2}}$. □

   (b) Fix a point $(u_0, v_0) \in U$, and let $\boldsymbol{\gamma}$ be the curve $t \mapsto (t, v_0, f(t, v_0))$ and let $\widetilde{\boldsymbol{\gamma}}$ be the curve $t \mapsto (u_0, t, f(u_0, t))$. Compute the normal curvatures of $\boldsymbol{\gamma}$ and $\widetilde{\boldsymbol{\gamma}}$. (Note: these curves are not unit-speed. See exercise 6.16 in Pressley for a formula for a general regular curve.)

   Solution: We'll also need the first fundamental form, which is easily computed from above. $E = 1 + f_u^2, F = f_u f_v$, and $G = 1 + f_v^2$. Then the exercise in Pressley says that

$$\kappa_n = \frac{L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2}{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}.$$

   We'll want to apply this to both $\boldsymbol{\gamma}$ and $\widetilde{\boldsymbol{\gamma}}$. For $\boldsymbol{\gamma}$, we have $u(t) = t$ and $v(t) = v_0$, and for $\widetilde{\boldsymbol{\gamma}}$ we have $\widetilde{u}(t) = u_0$ and $\widetilde{v}(t) = t$. So $\dot{u} = 1, \dot{v} = 0$ and $\dot{\widetilde{u}} = 0, \dot{\widetilde{v}} = 1$, and then if we let κ_n and $\widetilde{\kappa_n}$ denote the normal curvatures of $\boldsymbol{\gamma}$ and $\widetilde{\boldsymbol{\gamma}}$ respectively, then

$$\kappa_n = \frac{f_{uu}}{(1 + f_u^2)\sqrt{1 + f_u^2 + f_v^2}}$$

$$\widetilde{\kappa_n} = \frac{f_{vv}}{(1 + f_v^2)\sqrt{1 + f_u^2 + f_v^2}}.$$

□