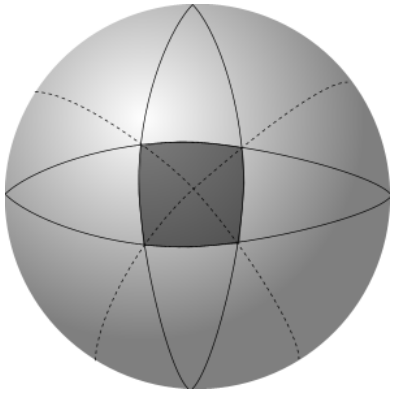
MATH 436
HOMEWORK 6
DUE OCTOBER 30, 2007

(1) Suppose that Q is a quadrilateral on the unit sphere S^2 whose sides are all (pieces of) great circles. Suppose that each of the two great-circle arcs connecting opposite angles of Q are in the interior of Q. An example is pictured below, with Q shaded.



Find a formula for the area of Q in terms of the angles of Q. Generalize to an arbitrary polygon.

(2) One possible smooth atlas for the unit cylinder $C = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 = 1\right\}$ follows: Let $U = (0, 2\pi) \times \mathbb{R} \subset \mathbb{R}^2$ and $U' = (-\pi, \pi) \times \mathbb{R} \subset \mathbb{R}^2$. Let $\boldsymbol{\sigma}\colon U \to \mathbb{R}^3$ be given by

$$\boldsymbol{\sigma}(u, v) = (\cos(u), \sin(u), v)\,,$$

and let $\boldsymbol{\sigma}'\colon U' \to \mathbb{R}^3$ be given by

$$\boldsymbol{\sigma}'(u, v) = (\cos(u), \sin(u), v)\,.$$

(It is intentional that they have the same formula.) Compute the second fundamental form for the patches in this atlas.

(3) Draw examples of curves with vanishing geodesic curvature on the following surfaces:
   (a) The sphere $\left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 = 1\right\}$.
   (b) The torus $\left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; (\sqrt{x^2 + y^2} - 3)^2 + z^2 = 1\right\}$.
   (c) The cylinder $\left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 = 1\right\}$.

(4) Let $U \subset \mathbb{R}^2$ and $f\colon U \to \mathbb{R}$ be a smooth map. Let

$$\Gamma(f) = \left\{(x, y, f(x, y)) \in \mathbb{R}^3 \;\middle|\; (x, y) \in U\right\}.$$

   (a) Compute the second fundamental form of (an appropriate regular surface patch of) $\Gamma(f)$.
   (b) Fix a point $(u_0, v_0) \in U$, and let $\boldsymbol{\gamma}$ be the curve $t \mapsto (t, v_0, f(t, v_0))$ and let $\widetilde{\boldsymbol{\gamma}}$ be the curve $t \mapsto (u_0, t, f(u_0, t))$. Compute the normal curvatures of $\boldsymbol{\gamma}$ and $\widetilde{\boldsymbol{\gamma}}$. (Note: these curves are not unit-speed. See exercise 6.16 in Pressley for a formula for a general regular curve.)

Date: October 23, 2007.