MATH 436
HOMEWORK 7
DUE NOVEMBER 5, 2007

SOLUTIONS

(1) Compute

$$\int_{S^2} K d\mathcal{A}$$

where S^2 is, as usual, the unit sphere in $\mathbb{R}^3$, and K is the Gaussian curvature. You can choose the atlas (though using angular spherical coordinates lets you get away with only doing one integral - you'll need to say why though).

Solution: This is particularly easy since $K = 1$ (constant) for the unit sphere S^2. So we're really just computing the area of the sphere, which is 4π. □

(2) Suppose a surface S is tangent to a plane P along a curve $\boldsymbol{\gamma}$. Prove that each point on the curve $\boldsymbol{\gamma}$ is either parabolic or planar.

Proof. We simply want to show that one of the principal curvatures is zero. Note that, along the curve $\boldsymbol{\gamma}$, the normal vector is constant. So if we look at $\boldsymbol{N}(t)$, the normal vector at $\boldsymbol{\gamma}(t)$, where $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ for some patch $\boldsymbol{\sigma}$, we know that $\dot{\boldsymbol{N}} = 0$. Now note that if $T = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a tangent vector, then $\begin{pmatrix} \boldsymbol{\sigma}_u & \boldsymbol{\sigma}_v \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$ if and only if $T = 0$ since $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ span the tangent plane. So

$$\begin{aligned} 0 &= \dot{\boldsymbol{N}} \\ &= \boldsymbol{N}_u \dot{u} + \boldsymbol{N}_v \dot{v} \\ &= \begin{pmatrix} \boldsymbol{N}_u & \boldsymbol{N}_v \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \\ &= -\begin{pmatrix} \boldsymbol{\sigma}_u & \boldsymbol{\sigma}_v \end{pmatrix} \mathcal{W} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}, \end{aligned}$$

hence $\mathcal{W} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = 0$. This means that $\dot{\boldsymbol{\gamma}}$ is an eigenvector of $\mathcal{W}$ with eigenvalue 0. But the eigenvalues of $\mathcal{W}$ are the principal curvatures. □

(3) Compute the Gaussian and Mean curvatures of the cylinder

$$\boldsymbol{\sigma}(u, v) = (\cos(u), \sin(u), v).$$

Date: October 30, 2007.

Solution: One can easily compute

$$\begin{aligned} E &= 1 \\ F &= 0 \\ G &= 1 \\ L &= -1 \\ M &= 0 \\ N &= 0. \end{aligned}$$

Then

$$\begin{aligned} K &= \frac{LN - M^2}{EG - F^2} \\ &= 0 \\ H &= \frac{LG - 2MF + NE}{2(EG - F^2)} \\ &= \frac{-1}{2}. \end{aligned}$$

□

(4) A curve $\mathcal{C}$ on a surface S is a *line of curvature* of S if the tangent vector to $\mathcal{C}$ is a principal vector of S at each point of $\mathcal{C}$. Draw some lines of curvature on
   (a) A sphere
   (b) A torus
   (c) A cylinder

Solution: For a sphere, every point is umbilic, so every tangent vector is a principal vector. So any old curve on S^2 will do. For a torus, we've shown before that the principal directions are $\boldsymbol{\sigma}_\theta$ and $\boldsymbol{\sigma}_\varphi$, so lines of curvature will be circles with fixed ϕ and circles with fixed θ on the torus. Similarly, for a cylinder the lines of curvature will be circles of constant z and lines of constant ϕ (in cylindrical coordinates). □

(5) Compute the principal curvatures of the patch

$$\boldsymbol{\sigma}(u, v) = \left(u, v, u^2 - v^2\right)$$

at the point $(0, 0, 0)$.

Solution:

$$
\begin{aligned}
\boldsymbol{\sigma}_u &= (1, 0, 2u) \\
\boldsymbol{\sigma}_v &= (0, 1, -2v) \\
\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (-2u, 2v, 1) \\
\boldsymbol{N} &= \left(\frac{-2u}{\sqrt{1+4u^2+4v^2}}, \frac{2v}{\sqrt{1+4u^2+4v^2}}, \frac{1}{\sqrt{1+4u^2+4v^2}} \right) \\
\boldsymbol{\sigma}_{uu} &= (0, 0, 2) \\
\boldsymbol{\sigma}_{uv} &= (0, 0, 0) \\
\boldsymbol{\sigma}_{vv} &= (0, 0, -2) \\
E &= 1 + 4u^2 \\
F &= -4uv \\
G &= 1 + 4v^2 \\
L &= \frac{2}{\sqrt{1+4u^2+4v^2}} \\
M &= 0 \\
N &= \frac{-2}{\sqrt{1+4u^2+4v^2}}.
\end{aligned}
$$

So the Gaussian and mean curvatures are

$$
\begin{aligned}
K &= \frac{\frac{-4}{1+4u^2+4v^2}}{1 + 4u^2 + 4v^2 + 16u^2v^2 - 16u^2v^2} \\
&= \frac{-4}{(1+4u^2+4v^2)^2} \\
H &= \frac{4\left(v^2 - u^2\right)}{\left(1 + 4\,v^2 + 4\,u^2\right)^{\frac{3}{2}}}
\end{aligned}
$$

Now using the formula that the principal curvatures are

$$H \pm \sqrt{H^2 - K},$$

and plugging in the point $(0, 0)$, we get

$$
\begin{aligned}
\kappa_1 &= 2 \\
\kappa_2 &= -2
\end{aligned}
$$

□