MATH 436
HOMEWORK 8
DUE NOVEMBER 13, 2007

SOLUTIONS

(1) Let
$$\boldsymbol{\sigma}(u, v) = (u, \cosh(u)\cos(v), \cosh(u)\sin(v)).$$
Compute the Gaussian and mean curvatures of $\boldsymbol{\sigma}$.

Solution: We basically went through this in class with the coordinates shuffled (this a catenoid). You should get
$$K = -\frac{1}{\cosh^4(u)}$$
$$H = 0.$$
□

(2) Suppose T is a torus in $\mathbb{R}^3$. Show there is a point $x \in T$ with Gaussian curvature $K < 0$ at x.

Proof. What I had in mind was that
$$\int_T K \, d\mathcal{A} = 0,$$
and the torus is compact hence it has a point with positive K, meaning it must have a point with negative K. We went through the proof in class for a specific torus that the integral above holds; read over that and note that the proof can be applied to any torus obtained by rotating a circle around the z–axis, and note that any torus is a coordinate change away from such a torus. □

(3) Suppose that $\boldsymbol{\gamma}$ is a curve on a surface S with normal curvature $\kappa_n = 0$. Show that the Gaussian curvature K of S satisfies $K \leq 0$ along $\boldsymbol{\gamma}$.

Proof. Since
$$\kappa_n = \kappa_1 \cos^2(\theta) + \kappa^2 \sin^2(\theta),$$
where the κ_i are the principal curvature, there are three possibilities (since both trig terms can't be equal to zero simultaneously):
(1) Both principal curvatures are equal to 0.
(2) One of the principal curvatures is zero, and the other trig term is zero.
(3) The trig terms are equal and nonzero, and the principal curvatures are nonzero, and equal in magnitude with opposite signs.

In the first two cases $K = 0$, and in the last case $K < 0$. □

(4) Let $\boldsymbol{\sigma}$ be a patch, and x a point on $\boldsymbol{\sigma}$. Suppose that the unit normal $\boldsymbol{N}$ at x satisfies $\boldsymbol{N}_u = \boldsymbol{\sigma}_v$ and $\boldsymbol{N}_v = -\boldsymbol{\sigma}_u$. Compute the Gaussian and mean curvatures of $\boldsymbol{\sigma}$ at x.

Date: November 6, 2007.

Solution: The Weingarten matrix can be read off from the equations for $\boldsymbol{N}_u$ and $\boldsymbol{N}_v$:

$$\mathcal{W} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Gaussian curvature is the determinant of $\mathcal{W}$ and the mean curvature is half the trace of $\mathcal{W}$.

$$\begin{aligned} K &= \det(\mathcal{W}) \\ &= 1 \\ H &= \frac{1}{2}\operatorname{tr}(\mathcal{W}) \\ &= 0. \end{aligned}$$

□