MATH 436
HOMEWORK 9
DUE DECEMBER 4, 2007

SOLUTIONS

(1) Suppose that $\boldsymbol{\sigma}(u,v)=\boldsymbol{\gamma}(u)+v\boldsymbol{\delta}(u)$ is a ruled surface (see Example 4.12 in the text). Show for any fixed u_0, the curve

$$\boldsymbol{\alpha}(t)=\boldsymbol{\gamma}(u_0)+t\boldsymbol{\delta}(u_0)$$

is a geodesic.

Solution: These are the rulings on the ruled surface, i.e. straight lines. Since any straight line is a geodesic, these are geodesics. □

(2) By a direct computation, show that the equator, parametrized as a unit-speed curve, on the unit sphere S^2 satisfies the geodesic equations, using any parametrization of S^2 you'd like.

Solutions: I'll use the patch with

$$\boldsymbol{\sigma}(\theta,\varphi)=(\sin(\theta)\cos(\varphi),\sin(\theta)\sin(\varphi),\cos(\theta)).$$

This misses a point on the equator, but I can change the φ domain to pick it up with the exact same formula. So let's just work in this patch. The equator has $\theta=\frac{\pi}{2}$, so the equator is

$$\gamma(t)=(\cos(t),\sin(t),0),$$

hence is $\boldsymbol{\sigma}(\theta(t),\varphi(t))$ with $\theta(t)=\frac{\pi}{2}$ and $\varphi(t)=t$. Also, we've computed several times that this patch has first fundamental form with $E=1,F=0,G=\sin^2(\theta)$. The geodesic equations are

$$\begin{aligned}\frac{d}{dt}\left(E\dot{\theta}+F\dot{\varphi}\right)&=\frac{1}{2}\left(E_\theta\dot{\theta}^2+2F_\theta\dot{\theta}\dot{\varphi}+G_\theta\dot{\varphi}^2\right)\\ \frac{d}{dt}\left(F\dot{\theta}+G\dot{\varphi}\right)&=\frac{1}{2}\left(E_\varphi\dot{\theta}^2+2F_\varphi\dot{\theta}\dot{\varphi}+G_\varphi\dot{\varphi}^2\right).\end{aligned}$$

Plugging in $\dot{\theta}=0,\dot{\varphi}=1,E_\theta=0,E_\varphi=0,F_\theta=0,F_\varphi=0,G_\theta=2\sin(\theta)\cos(\theta),G_\varphi=0$, and noting that since $\theta=\frac{\pi}{2},G_\theta=0$ and $\dot{G}=0$ since G is constant when θ is constant, both of these equations simplify down to $0=0$. □

(3) Use the corollary to the Theorema Egregium (Corollary 10.1 in the text) to compute the Gaussian curvature of the patch

$$\boldsymbol{\sigma}(\theta,\varphi)=(\sin(\theta)\cos(\varphi),\sin(\theta)\sin(\varphi),\cos(\theta)).$$

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Solution: We have $E = 1, F = 0, G = \sin^2(\theta)$. So all of the derivatives of E and F are zero, $G_\varphi = 0$, and $G_\theta = 2\sin(\theta)\cos(\theta) = \sin(2\theta)$, so $G_{\theta\theta} = 2\cos(2\theta)$. The the corollary to the Theorema Egregium (Corollary 10.1 in the text) says

$$\begin{aligned}
K &= \frac{\det\begin{pmatrix} -\frac{1}{2}E_{\varphi\varphi} + F_{\theta\varphi} - \frac{1}{2}G_{\theta\theta} & \frac{1}{2}E_\theta & F_\theta - \frac{1}{2}E_\varphi \\ F_\varphi - \frac{1}{2}G_\theta & E & F \\ \frac{1}{2}G_\varphi & F & G \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}E_\varphi & \frac{1}{2}G_\theta \\ \frac{1}{2}E_\varphi & E & F \\ \frac{1}{2}G_\theta & F & G \end{pmatrix}}{\left(EG - F^2\right)^2} \\
&= \frac{\det\begin{pmatrix} -\cos(2\theta) & 0 & 0 \\ -\sin(\theta)\cos(\theta) & 1 & 0 \\ 0 & 0 & \sin^2(\theta) \end{pmatrix} - \det\begin{pmatrix} 0 & 0 & \sin(\theta)\cos(\theta) \\ 0 & 1 & 0 \\ \sin(\theta)\cos(\theta) & 0 & \sin^2(\theta) \end{pmatrix}}{\sin^4(\theta)} \\
&= \frac{-\cos(2\theta)\sin^2\theta + \sin^2(\theta)\cos^2(\theta)}{\sin^4(\theta)} \\
&= \frac{\sin^2(\theta)\left(-\cos(2\theta) + \cos^2(\theta)\right)}{\sin^4(\theta)} \\
&= \frac{\sin^2(\theta)\left(\sin^2(\theta) - \cos^2(\theta) + \cos^2(\theta)\right)}{\sin^4(\theta)} \\
&= 1.
\end{aligned}$$

$\square$

(4) In the proof of Archimedes's Theorem (see Section 5.5 in the text if you need to refresh your memory) we constructed a conformal map from S^2 to a cylinder. Without computing what this map does to fundamental forms, use the Theorema Egregium to say why the map cannot be an isometry.

Solution: They don't have the same Gaussian curvature. $\square$