Name: Solution

QUIZ
DECEMBER 11, 2007

Most of these were just problems out of Pressley, so you can find solutions in the back.

(All Groups) (a) Find a parameterization for the level curve

$$y^2 - x^2 = 1.$$

This is Pressley exercise 1.2(i)

(b) Show that $\boldsymbol{\gamma}(t) = \left(\frac{4}{5}\cos(t), 1 - \sin(t), -\frac{3}{5}\cos(t)\right)$ is a unit speed curve. Compute its curvature and torsion.

This is Pressley exercise 2.1(ii) and 2.14(ii)

(c) Compute the first and second fundamental forms of the surface patch

$$\boldsymbol{\sigma}(u, v) = \left(u - v, u + v, u^2 + v^2\right).$$

The first fundamental form is Pressley exercise 5.1(ii).

$$\begin{aligned}
\boldsymbol{\sigma}_u &= (1, 1, 2u) \\
\boldsymbol{\sigma}_v &= (-1, 1, 2v) \\
\boldsymbol{N} &= \frac{1}{\sqrt{2(u^2+v^2)}}(v - u, -v - u, 1) \\
\boldsymbol{\sigma}_{uu} &= (0, 0, 2) \\
\boldsymbol{\sigma}_{uv} &= (0, 0, 0) \\
\boldsymbol{\sigma}_{vv} &= (0, 0, 2) \\
M &= 0 \\
L = N &= \sqrt{\frac{2}{u^2+v^2}}.
\end{aligned}$$

(d) Show if a curve on a surface has zero geodesic and normal curvature everywhere then it is part of a line.

Proof. Since $\kappa^2 = \kappa_n^2 + \kappa_g^2$, this means $\kappa = 0$, and any curve with everywhere vanishing curvature is part of a line. □

(Group 1) Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a smooth function such that f_x, f_y, and f_z don't all vanish at any point. Let $S = f^{-1}(0)$. Show

$$\boldsymbol{\nabla} f = (f_x, f_y, f_z)$$

is always perpendicular to S, and that S is orientable.

Proof. To show that $\boldsymbol{\nabla} f$ is perpendicular to S, we need to show it is perpendicular to any tangent vector of any curve on S. So suppose $\boldsymbol{\gamma}(t)$ is a curve on S. Then

$$\boldsymbol{\gamma}(t) = (x(t), y(t), z(t))$$

is a curve in $\mathbb{R}^3$ with $f(x(t), y(t), z(t)) = 0$. Then differentiating, we get

$$\begin{aligned} \frac{d}{dt} f(x(t), y(t), z(t)) &= 0 \\ f_x \dot{x} + f_y \dot{y} + f_z \dot{z} &= 0 \\ \boldsymbol{\nabla} f \cdot (\dot{x}, \dot{y}, \dot{z}) &= 0 \\ \boldsymbol{\nabla} f \cdot \dot{\boldsymbol{\gamma}} &= 0. \end{aligned}$$

So $\boldsymbol{\nabla} f$ is perpendicular to S. But then the vector $\boldsymbol{N} = \frac{\boldsymbol{\nabla} f}{\|\boldsymbol{\nabla} f\|}$ is a consistent choice of normal vector to the surface (note the definition of $\boldsymbol{N}$ makes sense since $\boldsymbol{\nabla} f$ is never zero). □

(Group 2) Let $\boldsymbol{\gamma}(s)$ be a unit speed curve in $\mathbb{R}^3$, with principal normal $\boldsymbol{n}$ and binormal $\boldsymbol{b}$. The *tube* of radius a around $\boldsymbol{\gamma}$ is parameterized by

$$\boldsymbol{\sigma}(s, \theta) = \boldsymbol{\gamma}(s) + a(\boldsymbol{n}(s)\cos(\theta) + \boldsymbol{b}(s)\sin(\theta)).$$

Prove that $\boldsymbol{\sigma}$ is regular if the curvature κ of $\boldsymbol{\gamma}$ is less than a^{-1} everywhere.
This is part of Pressley exercise 5.17.

(Group 3) Compute the geodesic curvature of any circle on the unit sphere (not necessarily a great circle).
This is Pressley exercise 6.8 (but also see 6.7).

(Group 4) A surface patch has first fundamental form

$$\cos^2(u)du^2 + \cosh^2(v)dv^2.$$

Compute the Christoffel symbols for this patch.

Solution.

$$\begin{aligned}
E &= \cos^2(u) \\
E_u &= -2\cos(u)\sin(u) \\
E_v &= 0 \\
F &= F_u = F_v = 0 \\
G &= \cosh^2(v) \\
G_u &= 0 \\
G_v &= 2\cosh(v)\sinh(v) \\
EG - F^2 &= \cos^2(u)\cosh^2(v) \\
\Gamma^1_{11} &= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)} \\
&= -2\tan(u) \\
\Gamma^2_{11} &= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)} \\
&= 0 \\
\Gamma^1_{12} &= \frac{GE_v - FG_u}{2(EG - F^2)} \\
&= 0 \\
\Gamma^2_{12} &= \frac{EG_u - FE_v}{2(EG - F^2)} \\
&= 0 \\
\Gamma^1_{22} &= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)} \\
&= 0 \\
\Gamma^2_{22} &= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)} \\
&= 2\tanh(v).
\end{aligned}$$

□