Topology/Geometry Qualifying Examination
August, 2002
Correction

This is a corrected version of the last problem on the August 2002 Qualifying exam. As it is stated on the exam, the problem is incorrect, and if you don't know what the Hopf map is, pretty unclear. The version here is correct and includes the relevant information on the Hopf map.

(6) Let $f : S^2 \to S^2$ be a map of degree $k > 1$ and let $h : S^3 \to S^2$ be the Hopf map (if $\mathbb{CP}^2$ is constructed as a CW complex with 1 0-cell, 1 2-cell, and 1 4-cell, the 4-cell is attached to the 2-skeleton, an S^2, by the Hopf map). Let X be the cell complex $e^0 \cup e^2 \cup \cup e^3 \cup e^4$ where the 2-cell e^2 is attached to the 0-cell by the constant map, the 3-cell is attached to the 2-skeleton $e^0 \cup e^2 \approx S^2$ by the map f of degree k, and the 4-cell e^4 is attached to the 2-skeleton $e^0 \cup e^2 \approx S^2$ by the Hopf map h.
Let Y be the cell complex $e^0 \cup e^2 \cup e^3 \cup e^4$ where the 2-cell e^2 is attached to the 0-cell by the constant map, the 3-cell is attached to the 2-skeleton $e^0 \cup e^2 \approx S^2$ by the map f of degree k, and the 4-cell e^4 is attached to the 0-cell by the constant map.

(a) Compute the homology and cohomology of X and Y with integer coefficients and show that $H^*(X;\mathbb{Z}) \cong H^*(Y;\mathbb{Z})$ as rings.

(b) Show that $H^*(X;\mathbb{Z}_k)$ and $H^*(Y;\mathbb{Z}_k)$ are isomorphic as **groups** but *not* isomorphic as **rings**.