Topology/Geometry Qualifying Examination
Practice Exam 1

1. Recall that if Z is a subspace of the topological set X, an *accumulation point* of Z in X is an element $x \in X$ such that for every open set U containing x, $(U - \{x\}) \cap Z$ is nonempty. Let $A \subset S^2$ be a set with no accumulation points in S^2.

   (a) Prove that A is a finite set.

   (b) Calculate the fundamental group $\pi_1(S^2 - A)$.

2. Define the (infinite) CW complex S^∞ as follows:
   For each $n \geq 0$, S^∞ has two n-cells, a_n and b_n. We then define the structure inductively: note that the 0-skeleton consists of two points, i.e. an S^0. The 1-cells are attached to the 0-skeleton as the upper and lower hemisphers of an S^1 with the 0-skeleton playing the part of the equator. We then continue, so that for each $k \geq 0$, the k-skeleton of S^∞ is just an S^k, and the $k+1-$cells are attached to the k-skeleton as the upper and lower hemispheres of a $k+1-$sphere with the k-skeleton playing the part of the equator in the $k+1-$skeleton.

   (a) Show that for any $k \geq 0$, any map $S^k \to S^\infty$ is nullhomotopic.

   (b) Show that there is a well defined antipodal map $S^\infty \to S^\infty$ that restricts to the normal antipodal map on each k-skeleton.

   (c) Define $\mathbb{RP}^\infty$ as the quotient of S^∞ by the antipodal map, i.e. antipodal points are identified. Compute the fundamental group of $\mathbb{RP}^\infty$.

3. Let $h : M \to N$ be a smooth map between smooth manifolds M, N, where $\dim(M) \leq \dim(N)$.

   (a) If M is compact, show that the set of regular values of h is an open, dense subset of N.

   (b) Show that if M is *not* compact then the set of regular values of h need not be open in N.

4. Suppose W is a compact Hausdorff space.

   (a) Show W is normal (*Hint: First show W is regular*).

   (b) Let $B \subset W$ be a closed subspace with $\pi_1(B)$ a non-abelian simple group (Recall a group is simple if and only if it has no nontrivial normal subgroups). Let $f : B \to T^n$ be a continuous map where $T^n = \overbrace{S^1 \times S^1 \times \cdots \times S^1}^{n\ times}$. Show that f extends to a continous map $\widetilde{f} : W \to T^n$.

5. Let C be a CW-complex with 3 0-cells, 8 1-cells, and 2 2-cells. Suppose that C is homotopy equivalent to a connected, closed (compact without boundary) 2-manifold S.

   (a) Show that S is not orientable.

   (b) Compute the homology groups of S.

   (c) Let S' be a connected, orientable double cover of S. Then S' is a connected, closed, oriented 2-manifold. Compute the genus of S'.

6. Let $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^2$ be a homomorphism. Then φ can be represented as a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{Z}$. Recall that φ is an isomorphism if and only if the determinant of this matrix is ± 1.

   (a) Show for any such φ, there is a smooth map $g^\varphi : T^2 \to T^2$, where $T^2 = S^1 \times S^1$ is the torus, with $g^\varphi_\sharp : \pi_1(T^2) \to \pi_1(T^2)$ equal to the homomorphism φ.

   (b) Show that the matrix representing φ has determinant ± 1 if and only if the map g^φ can be chosen to be a diffeomorphism.