Topology/Geometry Qualifying Examination
Practice Exam 2

1. Let $A = (0 \cup (1,2)) \times \mathbb{Z}$, $B = (0 \cup (1,2]) \times \mathbb{Z}$ topologized as a subspaces of $\mathbb{R}^2$.

   (a) Prove that A and B are *not* homeomorphic.

   (b) Find a continuous bijection $A \to B$.

2. Let $U \subset \mathbb{R}^3$ be defined as the complement of the z-axis. Let $g : U \to \mathbb{R}$ be defined by

$$g(x, y, z) = (\sqrt{x^2 + y^2} - 5)^2 + z^2$$

   (a) Show g is smooth on U and show that 1 is a regular value of g.

   (b) Let $T = g^{-1}(1)$. Let $G : T \to \mathbb{R}$ be defined by

$$G(x, y, z) = z$$

   Find the critical points of G in T and describe the inverse images (in T) of regular values of G.

3. (a) Let $D_4 = \langle a, b | a^2 = 1, b^4 = 1, aba^{-1} = b^3 \rangle$ be the dihedral group of order 8. Construct a connected, two-dimensional CW complex Z with $\pi_1(Z) \simeq D_4$.

   (b) Find both a regular covering space and a nonregular covering space of Z.

4. Let X be a path connected topological space; recall that the cone over X, denoted CX, is defined as $(X \times [0, 1])/(X \times \{0\})$, and the suspension of X, denoted ΣX, is defined as $CX/(X \times \{1\})$.

   (a) Prove that the cone CX is contractible.

   (b) Prove that for reduced homologies, we have $\widetilde{H}_p(\Sigma X)$ is isomorphic to $\widetilde{H}_{p-1}(X)$ for all $p \geq 0$.

   (c) Prove that for any Y, $f : X \to Y$ continuous, f induces a map $\Sigma f : \Sigma X \to \Sigma Y$ and that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{H}_p(\Sigma X) & \xrightarrow{\simeq} & \widetilde{H}_{p-1}(X) \\ \downarrow (\Sigma f)_* & & \downarrow f_* \\ \widetilde{H}_p(\Sigma Y) & \xrightarrow{\simeq} & \widetilde{H}_{p-1}(Y) \end{array}$$

   where the horizontal isomorphisms are given by the previous part.

5. Let C be the two-dimensional CW complex obtained by attaching a 2-cell to a circle by a map of degree 5.

   (a) Compute the fundamental group of C, the homology groups of C with $\mathbb{Z}$-coefficients, and the cohomology groups of C with $\mathbb{Z}_5$-coefficients.

   (b) Compute the cup product $H^1(C;\mathbb{Z}_5) \times H^1(C;\mathbb{Z}_5) \to H^2(C;\mathbb{Z}_5)$.

6. Everyone knows what a connected sum of two manifolds is so I won't confuse you with yet another definition. Let M be the connected sum $\mathbb{RP}^4 \# \mathbb{CP}^2$.

   (a) Compute the fundamental group and the homology groups of M.

   (b) Is M orientable? Why or why not?

   (c) Describe the universal cover $\widetilde{M}$ of M.

   (d) Compute the cohomology **ring** of $\widetilde{M}$.