Topology/Geometry Qualifying Examination
Practice Exam 3

1. Let $Y$ be a metric space with metric $d$ and $Z$ be a metric space with metric $\rho$. Define a function $D : Y \times Z \times Y \times Z \longrightarrow \mathbb{R}$ by $D((y_1, z_1), (y_2, z_2)) = \sqrt{d(y_1, y_2)^2 + \rho(z_1, z_2)^2}$.
   (a) Show $D$ is a metric on $Y \times Z$.
   (b) Show that the product topology on $Y \times Z$ (induced by the metric topologies on $Y$ and $Z$) is the same as the metric topology on $Y \times Z$ induced by $D$.

2. Let $T^n = \overbrace{S^1 \times S^1 \times \ldots \times S^1}^{n \text{ times}} \subset \mathbb{C}^n = \{(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \mathbb{C}^n \mid \theta_k \in \mathbb{R}, 1 \leq k \leq n\}$. Let $\Sigma : T^n \to \mathbb{C}$ denote the map $\Sigma(e^{i\theta_1}, \ldots, e^{i\theta_n}) = e^{i\theta_1} + e^{i\theta_2} + \cdots + e^{i\theta_n}$. Find the critical points of $\Sigma$ considered as a smooth map of real manifolds.

3. Let $M$ denote the closed Möbius band, let $\partial M$ denote the boundary circle, and let $\imath : \partial M \hookrightarrow M$ denote the inclusion.
   (a) Compute:
      i. $\imath_\sharp : \pi_1(\partial M) \to \pi_1(M)$
      ii. $\imath_* : H_1(\partial M) \to H_1(M)$
      iii. $\imath^* : H^1(M) \to H^1(\partial M)$
   (b) Show $\partial M$ is not a retract of $M$, i.e. there does not exist a continuous $r : M \to \partial M$ with $r \circ \imath = \mathrm{id}_{\partial M}$.

4. Let $X$ be a CW-complex, and for each $n$, let $X^{(n)}$ denote its $n$-skeleton.
   (a) Show $X$ is connected if and only if $X^{(1)}$ is connected.
   (b) Show the inclusion $X^{(1)} \hookrightarrow X$ induces a surjection $\pi_1(X^{(1)}) \to \pi_1(X)$.
   (c) Show the inclusion $X^{(2)} \hookrightarrow X$ induces an isomorphism $\pi_1(X^{(2)}) \to \pi_1(X)$.
   (d) Suppose $X$ is $m$-dimensional. Let $f : X^{(m-1)} \to S^k$ be any continuous map, where $k \geq m$. Show that $f$ extends to a map $\widetilde{f} : X \to S^k$.

5. Let $L : \coprod_{i=1}^{N} S^1_i \to S^3$ be a smooth embedding of $N \geq 1$ disjoint $S^1$'s into $S^3$. Let $W$ denote the smooth 3-manifold $S^3 - \text{im}(L)$.

   (a) Show $W$ has the homotopy type of a compact 3-manifold with boundary, where the boundary is a disjoint union of $N$ tori (*Hint:* Use the Tubular Neighborhood Theorem... Bigger Hint - you may assume the Normal Bundle of $\text{im}(L)$ is trivial, i.e. is diffeomorphic to $\text{im}(L) \times \mathbb{R}^2$).

   (b) Compute $H_*(W)$.

6. Show $S^3 \times S^4$ is not homotopy equivalent to $S^3 \vee S^4 \vee S^7$.