

Topology/Geometry Qualifying Examination
Practice Exam 4

1. Compute the homology groups of the open subset Ω of \mathbb{R}^4 obtained by removing the four coordinate axes.
2. Let M be a smooth, connected, oriented n -manifold, $\partial M = \emptyset$, and suppose that M may be smoothly embedded as a closed (in the subspace topology) subset of \mathbb{R}^{n+1} .
Prove or disprove: The homology groups of M are all finitely generated.
3. Let $M_k(\mathbb{R})$ denote the set of all $k \times k$ matrices over \mathbb{R} , topologized as \mathbb{R}^{k^2} . Let $SO(k, \mathbb{R}) = \{A \in M_k(\mathbb{R}) \mid A^T A = I, \det(A) = 1\}$ where I denotes the $k \times k$ identity matrix, and A^T denotes the transpose of the matrix A .
 - (a) Show $SO(k, \mathbb{R})$ is compact.
 - (b) Compute the fundamental group of $SO(2, \mathbb{R})$.
4. Compute the cohomology ring of $\frac{S^1 \times S^1 \times S^1}{S^1 \times \{*\} \times \{*\}}$.
5. Let $*$ denote any basepoint of $\mathbb{R}P^2$. Consider $\mathbb{R}P^2 \vee \mathbb{R}P^2$ as a subset of $\mathbb{R}P^2 \times \mathbb{R}P^2$ by $\mathbb{R}P^2 \vee \mathbb{R}P^2 = \mathbb{R}P^2 \times \{*\} \cup \{*\} \times \mathbb{R}P^2 \subset \mathbb{R}P^2 \times \mathbb{R}P^2$. Show $\mathbb{R}P^2 \vee \mathbb{R}P^2$ is not a retract of $\mathbb{R}P^2 \times \mathbb{R}P^2$; i.e. there does not exist a continuous $r : \mathbb{R}P^2 \times \mathbb{R}P^2 \rightarrow \mathbb{R}P^2 \vee \mathbb{R}P^2$ which restricts to the identity on $\mathbb{R}P^2 \vee \mathbb{R}P^2$.
6. Let m, ℓ be positive integers, and let $p_m, p_\ell : S^1 \rightarrow S^1$ be covers with m, ℓ sheets respectively.
 - (a) Show $p_m \times p_\ell : S^1 \times S^1 \rightarrow S^1 \times S^1$ is a cover, and compute the number of sheets.
 - (b) Calculate the Lefschetz number (over \mathbb{Z}) of $p_m \times p_\ell$.