Topology/Geometry Qualifying Examination
Practice Exam 5

1. Let M_1, M_2 be closed n-dimensional manifolds. We say M_1 and M_2 are *cobordant* if there exists an $n+1$-dimensional manifold W such that ∂W is homeomorphic to the disjoint union $M_1 \coprod M_2$.

   (a) Show the relation "M_1 is cobordant to M_2" is an equivalence relation on closed n-manifolds.

   (b) Show all closed oriented 2-manifolds are cobordant

2. Recall if $f : X \to Y$ is continuous, the *mapping cylinder* of f, denoted M_f, is formed as the quotient space $\frac{(X\times[0,1])\coprod Y}{\sim}$ where $(x, 1) \sim f(x)$.

   (a) Show that Y is a strong deformation retract of M_f.

   (b) Show that the inclusion $j : X = X \times \{0\} \hookrightarrow M_f$ is homotopic to the map $\tilde{f} = i \circ f$ where $i : Y \hookrightarrow M_f$ is the inclusion.

   (c) Suppose that f induces isomorphisms $f_* : H_p(X) \to H_p(Y)$ for all p. Show that M_f/X is acyclic (You may assume that $j : X \hookrightarrow M_f$ is a cofibration).

3. Let K be a topological space and $\{K_n\}_{n\geq 1}$ a family of subspaces with $K_1 \subset K_2 \subset K_3 \subset \ldots$ and $X = \bigcup K_n$. We say K has the weak topology with respect to the family $\{K_n\}$ if $C \subset K$ is closed in K if and only if $C \cap K_n$ is closed in K_n for all n.

   (a) Suppose $g : K \to L$ is a function and K has the weak topology with respect to the family $\{K_n\}$. Show g is continuous if and only if $g_n = g\big|_{K_n} : K_n \to L$ is continuous for all n.

   (b) Let $\overline{K_n} = \bigcup_{1\leq k\leq n} C_k$ where C_k is the circle of radius $\frac{1}{k}$ and center $(\frac{1}{k}, 0)$ in $\mathbb{R}^2$. Let $\overline{K} = \bigcup \overline{K_n}$. Prove that $\overline{K}$ *does not* have the weak topology with respect to the family $\{\overline{K_n}\}$.

   (c) Find a function $\overline{g} : \overline{K} \to \overline{L}$ where each $\overline{g_n}$ is continuous but $\overline{g}$ is not (you choose the space $\overline{L}$).

4. Let M be a closed oriented $p + q + 1$-dimensional manifold, and let $\varphi : S^p \times \mathbb{D}^{q+1} \to M$ be an embedding. Let $M_0 = M - \text{int}(\text{im}(\varphi))$, i.e. the complement of the interior of the image of φ.

   (a) What is the boundary of M_0?

   (b) Compute $H_*(M, M_0)$.

5. Let Σ_g denote a connected closed oriented 2-manifold of genus $g \geq 0$. Compute the fundamental group and all homology groups of the space resulting from deleting k points from Σ_g, where $k \geq 1$.

6. Compute the cohomology ring of $(\mathbb{CP}^4 \times \mathbb{CP}^4)\#\mathbb{CP}^8$