Topology/Geometry Qualifying Examination
Practice Exam 6

1. Demonstrate the following (unrelated) propositions about metric spaces:

   (a) Show a retract of a complete metric space is complete.

   (b) Show a proper map of metric spaces has closed image.

   (c) Prove or disprove: Any metric space is locally compact.

2. Answer or demonstrate the following (unrelated) propositions about complex projective space, $\mathbb{CP}^n$.

   (a) Show $\mathbb{CP}^n$ is a smooth $2n$-manifold, i.e. explicitly give charts and explain why the change-of-coordinate maps are smooth (you may assume $\mathbb{CP}^n$ is second countable and Hausdorff).

   (b) Let $m : \mathbb{CP}^\ell \to \mathbb{CP}^\ell$ be a map of degree 32. What is ℓ?

   (c) Suppose $k < n$. Show the map $(z_0, \ldots, z_k) \mapsto (z_0, \ldots, z_k, 0, 0, \ldots, 0)$ of $\mathbb{C}^{k+1} \to \mathbb{C}^{n+1}$ induces a map $\mathbb{CP}^k \to \mathbb{CP}^n$ which maps $\mathbb{CP}^k$ diffeomorphically onto its image, and compute the cohomology ring of $\mathbb{CP}^n/\mathbb{CP}^k$ (you may assume, if you know it, the cohomology ring of $\mathbb{CP}^n$).

3. Let M be a simply connected manifold.

   (a) Show M is orientable.

   (b) Suppose M is compact, 4-dimensional, and $\chi(M) = 3$. Compute $H_*(M)$ and $H^*(M \times \mathbb{RP}^2; \mathbb{Z}_2)$ (groups only, not the ring structure).

4. Let G be a group with presentation $\langle x_1, x_2, \ldots, x_n | r_1, r_2, \ldots, r_m \rangle$.

   (a) Show there is a (connected) compact CW-complex C with basepoint $*$ such that $\pi_1(C, *) \approx G$.

   (b) Let $g \in G$ and let $H \subset G$ be the subgroup generated by g. Let $1 \in S^1$ be the basepoint. Show there is a map $f : (S^1, 1) \to (C, *)$ such that $\text{im}(f_\sharp) \approx H$ (where $f_\sharp : \pi_1(S^1, 1) \to \pi_1(C, *)$ is the homomorphism induced by f).

5. Let $g : \mathbb{RP}^{2j} \to \mathbb{RP}^{2j}$ be continuous. Recall $H^*(\mathbb{RP}^n; \mathbb{Z}_2) \approx \mathbb{Z}_2[x]/(x^{n+1})$ where $0 \neq x \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$. Suppose $g^*(x^j) \neq 0$. Show g has a fixed point.

6. Let M be a smooth n-manifold. We say M has trivial tangent bundle if there exist $X_1, \ldots, X_n$ smooth vector fields on M so that $\{X_1(p), \ldots, X_n(p)\}$ is a basis for $T_p(M)$ for all $p \in M$.

   (a) Show S^1 has trivial tangent bundle.

   (b) Show if M has trivial tangent bundle then M is orientable.

   (c) Show that the converse is not necessarily true.