RESEARCH STATEMENT

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My current research is on the Turaev torsion of compact, oriented, connected three dimensional manifolds with nonempty boundary. The references given below are for the first appearance, to my knowledge, of a concept; however [5] and [6] are excellent surveys. The Turaev torsion (see [2] where it is called the Maximal Abelian Torsion) is defined for a finite CW complex X with Euler characteristic $\chi(X) = 0$, and takes values in the quotient ring $Q(\mathbb{Z}[H_1(X)])$ obtained by localizing $\mathbb{Z}[H_1(X)]$ at the (multiplicative) set of nonzerodivisors. The Turaev torsion is one of many topological torsions, all of which are based on an algebraic notion of the torsion of a finite acyclic complex over a field, with a distinguished basis in each dimension. The algebraic torsion of such a complex is defined as a certain alternating product of determinants of change of basis matrices. To define topological torsions, one looks for acyclic chain complexes to associate with a space; in the case of Reidemeister torsion, these come from twisted complexes associated to homomorphisms $\varphi : \mathbb{Z}[H_1(X)] \to F$ where F is a field, though there is some indeterminacy up to the action of $\pm H_1(X)$ on F resulting from not having a natural way to choose distinguished bases.

Turaev torsion specifies certain homomorphisms to use, and pieces the resulting Reidemeister torsions together to obtain an element in $Q(\mathbb{Z}[H_1(X)])$. There are also refinements based on the notions of "Euler structures" (see [4]) and "homology orientations" (see [3]), which remove some classical indeterminacies from the torsion definition. Briefly, an Euler structure is a way to define a distinguished basis up to order and orientation, and a homology orientation is a way to account for the sign indeterminacy inherent in ignoring order and orientation above. We will follow Turaev's notation - for e an Euler structure and ω a homology orientation, the refined Turaev torsion of X will be denoted $\tau(X, e, \omega)$. In [6], Turaev proves that for connected, compact three-manifolds and connected two dimensional complexes with Euler characteristic zero, if the first Betti number is greater than or equal to 2, then the torsion actually is in $\mathbb{Z}[H_1(X)] \subset Q(\mathbb{Z}[H_1(X)])$, and indeed often in powers of the augmentation ideal.

For the purposes of what follows, "manifold" will mean a compact, connected, oriented three-manifold unless otherwise specified. My research is mostly into the torsions of manifolds with nonvoid boundary. For such a manifold, one must require $\chi(\partial M) = 0$ so that $\chi(M) = 0$. One way to enforce this would be to require ∂M to consist entirely of tori, though it certainly isn't necessary. However, if ∂M is not all tori, then at least one boundary component must be a sphere. In this situation, the following is known:

Lemma 1. *Let M be a compact, connected, oriented three-manifold with $\partial M \neq \varnothing$. If ∂M is not composed entirely of tori, then for any Euler structure e and homology orientation ω,*

$$\tau(M, e, \omega) = 0$$

In [6], Turaev gives various results relating cohomology to torsion for closed manifolds. Part of my thesis extends these results to manifolds with nonempty boundary. Specifically, for a closed manifold C with first Betti number $b_1(C) \geq 3$, there is an alternating trilinear form $H^1(C) \times H^1(C) \times H^1(C) \to \mathbb{Z}$ given by $(a, b, c) \mapsto \langle a \cup b \cup c, [C] \rangle$ where $[C] \in H_3(C)$ is the orientation class, and there is a known algebraic "determinant" of such a form, which unfortunately does not extend in an obvious way to a manifold M with $\partial M \neq \varnothing$. The simple reason is that in that case $H^3(M) = 0$ so we don't just want to cup together three things from $H^1(M)$. So the next most obvious thing is to make at least one of the elements that we cup together come from $H^1(M, \partial M)$ since the orientation class $[M]$ is in $H_3(M, \partial M)$, though it is not obvious how many entries should come from $H^1(M, \partial M)$, or where to put them. My own research shows that the correct mapping turns out to be the map $H^1(M, \partial M) \times H^1(M) \times H^1(M) \to \mathbb{Z}$ (given by the exact formula from above), which is alternate in the last two variables. In the case $b_1(M) \geq 2$, I have shown that there is an algebraic "determinant" of such a form, and given a homology orientation ω of M, a sign-refined determinant (this is a slight departure from the closed case, where the determinant has a canonical sign). If we use f_M to denote the form as defined above, then the ω-refined determinant is denoted $\mathrm{Det}_\omega(f_M)$. This determinant, like the determinant of an alternate trilinear form, takes its values in the symmetric algebra S on $H_1(M)/\operatorname{Tors}(H_1(M))$, whereas the Turaev torsion takes values, in the case $b_1(M) \geq 2$, in $I^{b_1(M)-2}$ where I is the augmentation ideal of $\mathbb{Z}[H_1(M)]$. In [6], for the purposes of comparing the determinant and the torsion, Turaev utilizes a grading preserving map $q : S \to \bigoplus_{\ell \geq 0} I^\ell / I^{\ell+1}$, which I also use in my own work. Then, generalizing the methods Turaev uses to prove his own theorem relating torsion and the determinant, I show in my dissertation

Theorem 1. *Let M be a compact, oriented, connected three-manifold, with $\partial M \neq \varnothing$, and $n = b_1(M) \geq 2$. Let I be the augmentation ideal of $\mathbb{Z}[H_1(M)]$, and let*

$$f_M : H^1(M, \partial M) \times H^1(M) \times H^1(M) \to \mathbb{Z}$$

be given by $(a, b, c) \mapsto \langle a \cup b \cup c, [M] \rangle$, where $[M] \in H_3(M, \partial M)$ is the orientation class. Then for any Euler structure e and homology orientation ω, $\tau(M, e, \omega) \in I^{n-2}$ (by [6]), and

$$\tau(M, e, \omega) \bmod I^{n-1} = |\operatorname{Tors}(H_1(M)|\; q(\mathrm{Det}_\omega(f_M))$$

Turaev obtains various generalizations of this result in [6]; he proves a version modulo certain integers, a generalization to higher order Massey products for integral cohomology (see [1] for basic definitions), and a version of that generalization modulo certain integers. I have completed the generalization to integral Massey products for manifolds with boundary, the main results are just below. The basic definition of Massey products is fairly involved, but the important things to recall are:

(1) The order two Massey product is given by cup products
(2) The order $m+1$ Massey product is a well-defined element if all order k Massey products for $2 \leq k \leq m$ vanish

So if all cup products vanish, there is a well-defined Massey triple product, and if all Massey triple products vanish, there is a well-defined Massey quadruple product, and so on. For our purposes, we assume we have an integer $m \geq 1$ such that for all $2 \leq k \leq m$, all order k Massey products vanish (this condition is void if $m = 1$), so that for $u_1, \ldots, u_{m+1} \in H^1(M)$, there is a Massey product $(u_1, \ldots, u_{m+1}) \in H^2(M)$. Then we have a multilinear form

$$f_M : H^1(M, \partial M) \times \overbrace{H^1(M) \times \cdots \times H^1(M)}^{m+1 \text{ times}} \to \mathbb{Z}$$

given by $f(v, u_1, \ldots, u_{m+1}) = -\langle v \cup (u_1, \ldots, u_{m+1}), [M]\rangle$ which satisfies certain conditions which I show suffice to know that it has an ω-refined determinant (for any homology orientation ω) denoted $\mathrm{Det}_\omega(f_M)$, and using the same map q as in Theorem 1,

Theorem 2. *Let M be a compact connected oriented 3-manifold with $\partial M \neq \varnothing, \chi(M) = 0$, $n = b_1(M) \geq 2$, that satisfies the condition (on Massey products) given above. Then $\tau(M, e, \omega) \in I^{m(n-1)-1}$ and*

$$\tau(M, e, \omega) \bmod I^{m(n-1)} = |\operatorname{Tors}(H_1(M))|\; q(\mathrm{Det}_\omega(f_M)) \in I^{m(n-1)-1}/I^{m(n-1)}$$

For $m = 1$, this is precisely Theorem 1. I have also completed the version of Theorem 1 modulo certain integers, and am currently working on the generalization to Massey products modulo certain integers.

My immediate plans for the future are to compare my results to Turaev's using his formulas for "gluing" of solid tori along boundary components. By Lemma 1, in the interesting cases the manifolds that I deal with have all toral boundary components, so after enough gluings, I can obtain a closed manifold, where Turaev's results apply. This will require studying how the various cohomological determinants change under gluings (in [6] Turaev gives gluing formulas for torsion). In addition to Turaev torsion, I have a general interest in algebraic topology, especially the study of cell complexes. In particular, I am interested in L^2-cohomology and group actions on products of spheres.

References

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